SYNOPSIS OF SOLID GEOMETRY
V.V.PRASOLOV AND I.F.SHARYGIN

The book contains 560 problems in solid geometry with complete solutions and 60 simple problems as exersises (ca 260 pages, 118 drawings).

The authors are the leading Russian experts in elementary geometry, especially in *problems* of elementary geometry. I. F. Sharygin leads the geometric part of "Problems" section in the magazin *Mathematics in school* and V. V. Prasolov is a consultant on geometric problems in *Kvant* (nowadays known in English version: *Quantum*). Many of the original problems suggested by the authors have been published in these magazins and proposed at the Moscow, All-Union (National) and International Mathematical Olympiads and other math competitions.

The authors collected huge archives of geometric problems that include files of mathematical olympiads and problems from many books and articles, both new and old. The problems in solid geometry from these articles form the main body of the book. Some of the problems in this book are new, they are proposed here for the first time.

The literature on solid geometry is much scantier as compared with the literature on, say, plane geometry. There is no book which reflects with sufficient completeness the modern condition of solid geometry. The authors hope that their book will fill in this gap because it contains almost all the known problems in solid geometry whose level of difficulty is not much higher than the level of abilities of an inteliigent student of a secondary school.

The most complete and meticulous book on solid geometry is the well-known *Elementary Geometry* by Hadamard. But Hadamard's book is slightly old-fashioned: many new problems and theorems has been discovered since it has been published and the mathematical olympiads usually intrude into the topics that Hadamard's book did not touch. Besides, Hadamard's book is primarily a manual and only secondly a problem book, therefore, it contains too many simple teaching problems.

Solutions of many problems from the book by Prasolov and Sharygin can be found elsewhere but even for the known problems almost every solution is newly rewritten specially for this book. The solutions were thoroughly studied and the shortest and most natural ways have been selected; the geometric (synthetic) methods were preferred. Sometimes (not often) the solutions are just sketched but all

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the essential points of the proofs are always indicated and only the absolutely clear details are omitted.

A characteristic feature of the book is a detailed classification of problems according to themes and methods of their solution. The problems are divided into 16 chapters subdivided into 5 or 6 sections each. The problems are arranged inside of each section in order of increasing level of difficulty. This stratification, not universally accepted, seem to be useful and helpful for the following reasons:

• *For the student*, solving problems that have similar ways of solutions helps to better absorb the topics; the headings help the student to find a way in the new subject and, to an extent, hint as to how to solve the problems.

• *For the teacher*, headings help to find problems that are connected with the topic needed; often headings help to recognize a problem and its solution as quickly as possible.

• The partitioning of the text helps to read the book, psychologically; especially so if the reader wants to read only a part of it.

The introductory part "Encounters with solid geometry" is of great interest. It contains problems with solutions that do not require any knowledge of solid geometry but a good spatial imagination.

The reviewrs of the first Russian edition mentioned a large spectrum of topics and the completeness of the book; thouroughly thought over and carefully presented laconic solutions and a helpful classification of problems according to their themes and methods of solution. Here are some excerpts:

N.B.Vasiliev: "The book suggested for publication in *Nauka*'s series *The School Mathematical Circle's Library* contains a rich collection of problems in solid geometry. It begins with teaching problems, a little more difficult than the usual highschool problems, and goes further to fully reflect problems and topics of mathematical olympiads usually studied at math circles.

The authors are well known by their bestsellers published by *Nauka* and translated by *Mir* in English and Spanish[1]. This book will certainly be met by the readers with great interest and will be useful for the students as well as for the teachers and the students of paedagogical departments of universities[2]. "

N. P. Dolbilin: "The spectrum of solid geometry in the book by Prasolov and Sharygin has encyclopaedical quality. ... of high importance methodologically is

[1] I.F.Sharygin, *Problems on plane geometry*, 1st ed. 1982, 2nd ed. 1986 (300 000 copies in Russian only); I. F. Sharygin, *Problems on solid geometry*, 1st ed. 1984 (150 000 copies in Russian only); V. V. Prasolov, *Problems on plane geometry* in 2 volumes, 1st ed. 1986 (400 000 copies in Russian); 2nd ed. 1989 (500 000 copies).

[2] Which proved to be the case. D.L.

the classification of problems according to their topics, methods of solution and level of difficulty".

CHAPTER 1. LINES AND PLANES IN SPACE

§1. Angles and distances between skew lines

1.1. Given cube $ABCDA_1B_1C_1D_1$ with side a. Find the angle and the distance between lines A_1B and AC_1.

1.2. Given cube with side 1. Find the angle and the distance between skew diagonals of two of its neighbouring faces.

1.3. Let K, L and M be the midpoints of edges AD, A_1B_1 and CC_1 of the cube $ABCDA_1B_1C_1D_1$. Prove that triangle KLM is an equilateral one and its center coincides with the center of the cube.

1.4. Given cube $ABCDA_1B_1C_1D_1$ with side 1, let K be the midpoint of edge DD_1. Find the angle and the distance between lines CK and A_1D.

1.5. Edge CD of tetrahedron $ABCD$ is perpendicular to plane ABC; M is the midpoint of DB, N is the midpoint of AB and point K divides edge CD in relation $CK : KD = 1 : 2$. Prove that line CN is equidistant from lines AM and BK.

1.6. Find the distance between two skew medians of the faces of a regular tetrahedron with edge 1. (Investigate all the possible positions of medians.)

§2. Angles between lines and planes

1.7. A plane is given by equation

$$ax + by + cz + d = 0.$$

Prove that vector (a, b, c) is perpendicular to this plane.

1.8. Find the cosine of the angle between vectors with coordinates (a_1, b_1, c_1) and (a_2, b_2, c_2).

1.9. In rectangular parallelepiped $ABCDA_1B_1C_1D_1$ the lengths of edges are known: $AB = a$, $AD = b$, $AA_1 = c$.

a) Find the angle between planes BB_1D and ABC_1.

b) Find the angle between planes AB_1D_1 and A_1C_1D.

c) Find the angle between line BD_1 and plane A_1BD.

1.10. The base of a regular triangular prism is triangle ABC with side a. On the lateral edges points A_1, B_1 and C_1 are taken so that the distances from them to the plane of the base are equal to $\frac{1}{2}a$, a and $\frac{3}{2}a$, respectively. Find the angle between planes ABC and $A_1B_1C_1$.

§3. Lines forming equal angles with lines and with planes

1.11. Line l constitutes equal angles with two intersecting lines l_1 and l_2 and is not perpendicular to plane Π that contains these lines. Prove that the projection of l to plane Π also constitutes equal angles with lines l_1 and l_2.

1.12. Prove that line l forms equal angles with two intersecting lines if and only if it is perpendicular to one of the two bisectors of the angles between these lines.

1.13. Given two skew lines l_1 and l_2; points O_1 and A_1 are taken on l_1; points O_2 and A_2 are taken on l_2 so that O_1O_2 is the common perpendicular to lines l_1 and l_2 and line A_1A_2 forms equal angles with linels l_1 and l_2. Prove that $O_1A_1 = O_2A_2$.

1.14. Points A_1 and A_2 belong to planes Π_1 and Π_2, respectively, and line l is the intersection line of Π_1 and Π_2. Prove that line A_1A_2 forms equal angles with planes Π_1 and Π_2 if and only if points A_1 and A_2 are equidistant from line l.

1.15. Prove that the line forming pairwise equal angles with three pairwise intersecting lines that lie in plane Π is perpendicular to Π.

1.16. Given three lines non-parallel to one plane prove that there exists a line forming equal angles with them; moreover, through any point one can draw exactly four such lines.

§4. Skew lines

1.17. Given two skew lines prove that there exists a unique segment perpendicular to them and with the endpoints on these lines.

1.18. In space, there are given two skew lines l_1 and l_2 and point O not on any of them. Does there always exist a line passing through O and intersecting both given lines? Can there be two such lines?

1.19. In space, there are given three pairwise skew lines. Prove that there exists a unique parallelepiped three edges of which lie on these lines.

1.20. On the common perpendicular to skew lines p and q, a point, A, is taken. Along line p point M is moving and N is the projection of M to q. Prove that all the planes AMN have a common line.

§5. Pythagoras's theorem in space

1.21. Line l constitutes angles α, β and γ with three pairwise perpendicular lines. Prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

1.22. Plane angles at the vertex D of tetrahedron $ABCD$ are right ones. Prove that the sum of squares of areas of the three rectangular faces of the tetrahedron is equal to the square of the area of face ABC.

1.23. Inside a ball of radius R, consider point A at distance a from the center of the ball. Through A three pairwise perpendicular chords are drawn.

a) Find the sum of squares of lengths of these chords.

b) Find the sum of squares of lengths of segments of chords into which point A divides them.

1.24. Prove that the sum of squared lengths of the projections of the cube's edges to any plane is equal to $8a^2$, where a is the length of the cube's edge.

1.25. Consider a regular tetrahedron. Prove that the sum of squared lengths of the projections of the tetrahedron's edges to any plane is equal to $4a^2$, where a is the length of an edge of the tetrahedron.

1.26. Given a regular tetrahedron with edge a. Prove that the sum of squared lengths of the projections (to any plane) of segments connecting the center of the tetrahedron with its vertices is equal to a^2.

§6. The coordinate method

1.27. Prove that the distance from the point with coordinates (x_0, y_0, z_0) to the plane given by equation $ax + by + cz + d = 0$ is equal to

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

1.28. Given two points A and B and a positive number $k \neq 1$ find the locus of points M such that $AM : BM = k$.

1.29. Find the locus of points X such that

$$pAX^2 + qBX^2 + rCX^2 = d,$$

where A, B and C are given points, p, q, r and d are given numbers such that $p + q + r = 0$.

1.30. Given two cones with equal angles between the axis and the generator. Let their axes be parallel. Prove that all the intersection points of the surfaces of these cones lie in one plane.

1.31. Given cube $ABCDA_1B_1C_1D_1$ with edge a, prove that the distance from any point in space to one of the lines AA_1, B_1C_1, CD is not shorter than $\frac{a}{\sqrt{2}}$.

1.32. On three mutually perpendicular lines that intersect at point O, points A, B and C equidistant from O are fixed. Let l be an arbitrary line passing through O. Let points A_1, B_1 and C_1 be symmetric through l to A, B and C, respectively. The planes passing through points A_1, B_1 and C_1 perpendicularly to lines OA, OB and OC, respectively, intersect at point M. Find the locus of points M.

Problems for independent study

1.33. Parallel lines l_1 and l_2 lie in two planes that intersect along line l. Prove that $l_1 \parallel l$.

1.34. Given three pairwise skew lines. Prove that there exist infinitely many lines each of which intersects all the three of these lines.

1.35. Triangles ABC and $A_1B_1C_1$ do not lie in one plane and lines AB and A_1B_1, AC and A_1C_1, BC and B_1C_1 are pairwise skew.

a) Prove that the intersection points of the indicated lines lie on one line.

b) Prove that lines AA_1, BB_1 and CC_1 either intersect at one point or are parallel.

1.36. Given several lines in space so that any two of them intersect. Prove that either all of them lie in one plane or all of them pass through one point.

1.37. In rectangular parallelepiped $ABCDA_1B_1C_1D_1$ diagonal AC_1 is perpendicular to plane A_1BD. Prove that this paral1lelepiped is a cube.

1.38. For which dispositions of a dihedral angle and a plane that intersects it we get as a section an angle that is intersected along its bisector by the bisector plane of the dihedral angle?

1.39. Prove that the sum of angles that a line constitutes with two perpendicular planes does not exceed 90°.

1.40. In a regular quadrangular pyramid the angle between a lateral edge and the plane of its base is equal to the angle between a lateral edge and the plane of a lateral face that does not contain this edge. Find this angle.

1.41. Through edge AA_1 of cube $ABCDA_1B_1C_1D_1$ a plane that forms equal angles with lines BC and B_1D is drawn. Find these angles.

Solutions

1.1. It is easy to verify that triangle A_1BD is an equilateral one. Moreover, point A is equidistant from its vertices. Therefore, its projection is the center of the triangle. Similarly, The projection maps point C_1 into the center of triangle A_1BD. Therefore, lines A_1B and AC_1 are perpendicular and the distance between them is equal to the distance from the center of triangle A_1BD to its side. Since all the sides of this triangle are equal to $a\sqrt{2}$, the distance in question is equal to $\frac{a}{\sqrt{6}}$.

1.2. Let us consider diagonals AB_1 and BD of cube $ABCDA_1B_1C_1D_1$. Since $B_1D_1 \parallel BD$, the angle between diagonals AB_1 and BD is equal to $\angle AB_1D_1$. But triangle AB_1D_1 is an equilateral one and, therefore, $\angle AB_1D_1 = 60°$.

It is easy to verify that line BD is perpendicular to plane ACA_1C_1; therefore, the projection to the plane maps BD into the midpoint M of segment AC. Similarly, point B_1 is mapped under this projection into the midpoint N of segment A_1C_1. Therefore, the distance between lines AB_1 and BD is equal to the distance from point M to line AN.

If the legs of a right triangle are equal to a and b and its hypothenuse is equal to c, then the distance from the vertex of the right angle to the hypothenuse is equal to $\frac{ab}{c}$. In right triangle AMN legs are equal to 1 and $\frac{1}{\sqrt{2}}$; therefore, its hypothenuse is equal to $\sqrt{\frac{3}{2}}$ and the distance in question is equal to $\frac{1}{\sqrt{3}}$.

1.3. Let O be the center of the cube. Then $2\{OK\} = \{C_1D\}$, $2\{OL\} = \{DA_1\}$ and $2\{OM\} = \{A_1C_1\}$. Since triangle C_1DA_1 is an equilateral one, triangle KLM is also an equilateral one and O is its center.

1.4. First, let us calculate the value of the angle. Let M be the midpoint of edge BB_1. Then $A_1M \parallel KC$ and, therefore, the angle between lines CK and A_1D is equal to angle MA_1D. This angle can be computed with the help of the law of cosines, because $A_1D = \sqrt{2}$, $A_1M = \frac{\sqrt{5}}{2}$ and $DM = \frac{3}{2}$. After simple calculations we get $\cos MA_1D = \frac{1}{\sqrt{10}}$.

To compute the distance between lines CK and A_1D, let us take their projections to the plane passing through edges AB and C_1D_1. This projection sends line A_1D into the midpoint O of segment AD_1 and points C and K into the midpoint Q of segment BC_1 and the midpoint P of segment OD_1, respectively.

The distance between lines CK and A_1D is equal to the distance from point O to line PQ. Legs OP and OQ of right triangle OPQ are equal to $\frac{1}{\sqrt{8}}$ and 1, respectively. Therefore, the hypothenuse of this triangle is equal to $\frac{3}{\sqrt{8}}$. The required distance is equal to the product of the legs' lengths divided by the length of the hypothenuse, i.e., it is equal to $\frac{1}{3}$.

1.5. Consider the projection to the plane perpendicular to line CN. Denote by X_1 the projection of any point X. The distance from line CN to line AM (resp. BK) is equal to the distance from point C_1 to line A_1M_1 (resp. B_1K_1). Clearly, triangle $A_1D_1B_1$ is an equilateral one, K_1 is the intersection point of its medians,

C_1 is the midpoint of A_1B_1 and M_1 is the midpoint of B_1D_1. Therefore, lines A_1M_1 and B_1K_1 contain medians of an isosceles triangle and, therefore, point C_1 is equidistant from them.

1.6. Let $ABCD$ be a given regular tetrahedron, K the midpoint of AB, M the midpoint of AC. Consider projection to the plane perpendicular to face ABC and passing through edge AB. Let D_1 be the projection of D, M_1 the projection of M, i.e., the midpoint of segment AK. The distance between lines CK and DM is equal to the distance from point K to line D_1M_1.

In right triangle D_1M_1K, leg KM_1 is equal to $\frac{1}{4}$ and leg D_1M_1 is equal to the height of tetrahedron $ABCD$, i.e., it is equal to $\sqrt{\frac{2}{3}}$. Therefore, the hypothenuse is equal to $\sqrt{\frac{35}{48}}$ and, finally, the distance to be found is equal to $\sqrt{\frac{2}{35}}$.

If N is the midpoint of edge CD, then to find the distance between medians CK and BN we can consider the projection to the same plane as in the preceding case. Let N_1 be the projection of point N, i.e., the midpoint of segment D_1K. In right triangle BN_1K, leg KB is equal to $\frac{1}{2}$ and leg KN_1 is equal to $\sqrt{\frac{1}{6}}$. Therefore, the length of the hypothenuse is equal to $\sqrt{\frac{5}{12}}$ and the required distance is equal to $\sqrt{\frac{1}{10}}$.

1.7. Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be points of the given plane. Then

$$ax_1 + by_1 + cz_1 - (ax_2 + by_2 + cz_2) = 0$$

and, therefore, $(x_1-x_2, y_1-y_2, z_1-z_2)\ perp(a, b, c)$. Consequently, any line passing through two points of the given plane is perpendicular to vector (a, b, c).

1.8. Since $(\mathbf{u}, \mathbf{v}) = |\mathbf{u}| \cdot |\mathbf{v}| \cos\varphi$, where φ is the angle between vectors $\mathbf{u}$ and $\mathbf{v}$, the cosine to be found is equal to

$$\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

1.9. a) **First solution**. Take point A as the origin and direct axes Ox, Oy and Oz along rays AB, AD and AA_1, respectively. Then the vector with coordinates $(b, a, 0)$ is perpendicular to plane BB_1D and vector $(0, c, -b)$ is perpendicular to plane ABC_1. Therefore, the cosine of the angle between given planes is equal to

$$\frac{ac}{\sqrt{a^2 + b^2} \cdot \sqrt{b^2 + c^2}}.$$

Second solution. If the area of parallelogram ABC_1D_1 is equal to S and the area of its projection to plane BB_1D is equal to s, then the cosine of the angle between the considered planes is equal to $\frac{s}{S}$ (see Problem 2.13). Let M and N be the projections of points A and C_1 to plane BB_1D. Parallelogram $MBND_1$ is the projection of parallelogram ABC_1D_1 to this plane. Since $MB = \frac{a^2}{\sqrt{a^2+b^2}}$, it follows that $s = \frac{a^2c}{\sqrt{a^2+b^2}}$. It remains to observe that $S = a\sqrt{b^2 + c^2}$.

b) Let us introduce the coordinate system as in the first solution of heading a). If the plane is given by equation

$$px + qy + rz = s,$$

then vector (p, q, r) is perpendicular to it. Plane AB_1D_1 contains points A, B_1 and D_1 with coordinates $(0, 0, 0)$, $(a, 0, c)$ and $(0, b, c)$, respectively. These conditions make it possible to find its equation:

$$bcx + acy - abz = 0;$$

hence, vector $(bc, ac, -ab)$ is perpendicular to the plane. Taking into account that points with coordinates $(0, 0, c)$, (a, b, c) and $(0, b, 0)$ belong to plane A_1C_1D, we find its equation and deduce that vector $(bc, -ac, -ab)$ is perpendicular to it. Therefore, the cosine of the angle between the given planes is equal to the cosine of the angle between these two vectors, i.e., it is equal to

$$\frac{a^2b^2 + b^2c^2 - a^2c^2}{a^2b^2 + b^2c^2 + a^2c^2}.$$

c) Let us introduce the coordinate system as in the first solution of heading a). Then plane A_1BD is given by equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

and, therefore, vector $abc(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}) = (bc, ca, ab)$ is perpendicular to this plane. The coordinates of vector $\{BD_1\}$ are $(-a, b, c)$. Therefore, the sine of the angle between line BD_1 and plane A_1BD is equal to the cosine of the angle between vectors $(-a, b, c)$ and (bc, ca, ab), i.e., it is equal to

$$\frac{abc}{\sqrt{a^2b^2c^2} \cdot \sqrt{a^2b^2 + b^2c^2 + c^2a^2}}.$$

1.10. Let O be the intersection point of lines AB and A_1B_1, M the intersection point of lines AC and A_1C_1. First, let us prove that $MO \perp OA$. To this end on segments BB_1 and CC_1 take points B_2 and C_2, respectively, so that $BB_2 = CC_2 = AA_1$. Clearly, $MA : AA_1 = AC : C_1C_2 = 1$ and $OA : AA_1 = AB : B_1B_2 = 2$. Hence, $MA : OA = 1 : 2$. Moreover, $\angle MAO = 60^\circ$ and, therefore, $\angle OMA = 90^\circ$. It follows that plane AMA_1 is perpendicular to line MO along which planes ABC and $A_1B_1C_1$ intersect. Therefore, the angle between these planes is equal to angle AMA_1 which is equal 45°.

1.11. It suffices to carry out the proof for the case when line l passes through the intersection point O of lines l_1 and l_2. Let A be a point on line l distinct from O; P the projection of point A to plane Π; B_1 and B_2 bases of perpendiculars dropped from point A to lines l_1 and l_2, respectively. Since $\angle AOB_1 = \angle AOB_2$, the right triangles AOB_1 and AOB_2 are equal and, therefore, $OB_1 = OB_2$. By the theorem on three perpendiculars $PB_1 \perp OB_1$ and $PB_2 \perp OB_2$. Right triangles POB_1 and POB_2 have a common hypothenuse and equal legs OB_1 and OB_2; hence, they are equal and, therefore, $\angle POB_1 = \angle POB_2$.

1.12. Let Π be the plane containing the given lines. The case when $l \perp \Pi$ is obvious. If line l is not perpendicular to plane Π, then l constitutes equal angles with the given lines if and only if its projection to Π is the bisector of one of the angles between them (see Problem 1.11); this means that l is perpendicular to another bisector.

1.13.Through point O_2, draw line l_1' parallel to l_1. Let Π be the plane containing lines l_2 and l_1'; A_1' the projection of point A_1 to plane Π. As follows from Problem 1.11, line $A_1'A_2$ constitutes equal angles with lines l_1' and l_2 and, therefore, triangle $A_1'O_2A_2$ is an equilateral one, hence, $O_2A_2 = O_2A_1' = O_1A_1$.

It is easy to verify that the opposite is also true: if $O_1A_1 = O_2A_2$, then line A_1A_2 forms equal angles with lines l_1 and l_2.

1.14. Consider the projection to plane Π which is perpendicular to line l. This projection sends points A_1 and A_2 into A_1' and A_2', line l into point L and planes Π_1 and Π_1 into lines p_1 and p_2, respectively. As follows from the solution of Problem 1.11, line A_1A_2 forms equal angles with perpendiculars to planes Π_1 and Π_2 if and only if line $A_1'A_2'$ forms equal angles with perpendiculars to lines p_1 and p_2, i.e., it forms equal angles with lines p_1 and p_2 themselves; this, in turn, means that $A_1'L = A_2'L$.

1.15. If the line is not perpendicular to plane Π and forms equal angles with two intersecting lines in this plane, then (by Problem 1.12) its projection to plane Π is parallel to the bisector of one of the two angles formed by these lines. We may assume that all the three lines meet at one point. If line l is the bisector of the angle between lines l_1 and l_2, then l_1 and l_2 are symmetric through l; hence, l cannot be the bisector of the angle between lines l_1 and l_3.

1.16. We may assume that the given lines pass through one point. Let a_1 and a_2 be the bisectors of the angles between the first and the second line, b_1 and b_2 the bisectors between the second and the third lines. A line forms equal angles with the three given lines if and only if it is perpendicular to lines a_i and b_j (Problem 1.12), i.e., is perpendicular to the plane containing lines a_i and b_j. There are exactly 4 distinct pairs (a_i, b_j). All the planes determined by these pairs of lines are distinct, because line a_i cannot lie in the plane containing b_1 and b_2.

1.17. First solution. Let line l be perpendicular to given lines l_1 and l_2. Through line l_1 draw the plane parallel to l. The intersection point of this plane with line l_2 is one of the endpoints of the desired segment.

Second solution. Consider the projection of given lines to the plane parallel to them. The endpoints of the required segment are points whose projections is the intersection point of the projections of given lines.

1.18. Let line l pass through point O and intersect lines l_1 and l_2. Consider planes Π_1 and Π_2 containing point O and lines l_1 and l_2, respectively. Line l belongs to both planes, Π_1 and Π_2. Planes Π_1 and Π_2 are not parallel since they have a common point, O; it is also clear that they do not coincide. Therefore, the intersection of planes Π_1 and Π_2 is a line. If this line is not parallel to either line l_1 or line l_2, then it is the desired line; otherwise, the desired line does not exist.

1.19. To get the desired parallelepiped we have to draw through each of the given lines two planes: a plane parallel to one of the remaining lines and a plane parallel to the other of the remaining lines.

1.20. Let PQ be the common perpendicular to lines p and q, let points P and Q belong to lines p and q, respectively. Through points P and Q draw lines q' and p' parallel to lines q and p. Let M' and N' be the projections of points M and N to lines p' and q'; let M_1, N_1 and X be the respective intersection points of planes passing through point A parallel lines p and q with sides MM' and NN' of the parallelogram $MM'NN'$ and with its diagonal MN (Fig. 16).

By the theorem on three perpendiculars $M'N \perp q$; hence, $\angle M_1N_1A = 90°$. It is

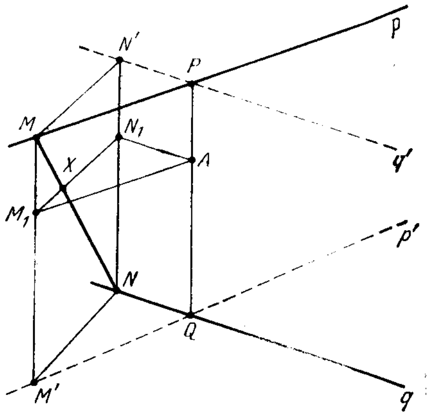

FIGURE 16 (SOL. 1.20)

also clear that

$$M_1X : N_1X = MX : NX = PA : QA;$$

therefore, point X belongs to a fixed line.

1.21. Let us introduce a coordinate system directing its axes parallel to the three given perpendicular lines. On line l take a unit vector $\mathbf{v}$. The coordinates of $\mathbf{v}$ are (x, y, z), where $x = \pm\cos\alpha$, $y = \pm\cos\beta$, $z = \pm\cos\gamma$. Therefore,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = x^2 + y^2 + z^2 = |\mathbf{v}|^2 = 1.$$

1.22. First solution. Let α, β and γ be angles between plane ABC and planes DBC, DAC and DAB, respectively. If the area of face ABC is equal to S, then the areas of faces DBC, DAC and DAB are equal to $S\cos\alpha$, $S\cos\beta$ and $S\cos\gamma$, respectively (see Problem 2.13). It remains to verify that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Since the angles α, β and γ are equal to angles between the line perpendicular to face ABC and lines DA, DB and DC, respectively, it follows that we can make use of the result of Problem 1.21.

Second solution. Let α be the angle between planes ABC and DBC; D' the projection of point D to plane ABC. Then $S_{DBC} = \cos\alpha S_{ABC}$ and $S_{D'BC} = \cos\alpha S_{DBC}$ (see Problem 2.13) and, therefore, $\cos\alpha = \frac{S_{DBC}}{S_{ABC}}$, $S_{D'BC} = \frac{S^2_{DBC}}{S_{ABC}}$ (Similar equalities can be also obtained for triangles $D'AB$ and $D'AC$). Taking the sum of the equations and taking into account that the sum of areas of triangles $D'BC$, $D'AC$ and $D'AB$ is equal to the area of triangle ABC we get the desired statement.

1.23. Let us consider the right parallelepiped whose edges are parallel to the given chords and points A and the center, O, of the ball are its opposite vertices. Let a_1, a_2 and a_3 be the lengths of its edges; clearly, $a_1^2 + a_2^2 + a_3^2 = a^2$.

a) If the distance from the center of the ball to the chord is equal to x, then the square of the chord's length is equal to $4R^2 - 4x^2$. Since the distances from the given chords to point O are equal to the lengths of the diagonals of parallelepiped's faces, the desired sum of squares is equal to

$$12R^2 - 4(a_2^2 + a_3^2) - 4(a_1^2 + a_2^2) - 4(a_1^2 + a_2^2) = 12R^2 - 8a^2.$$

b) If the length of the chord is equal to d and the distance between point A and the center of the chord is equal to y, the sum of the squared lengths of the chord's segments into which point A divides it is equal to $2y^2 + \frac{d^2}{2}$. Since the distances from point A to the midpoints of the given chords are equal to a_1, a_2 and a_3 and the sum of the squares of the lengths of chords is equal to $12R^2 - 8a^2$, it follows that the desired sum of the squares is equal to

$$2a^2 + (6R^2 - 4a^2) = 6R^2 - 2a^2.$$

1.24. Let α, β and γ be the angles between edges of the cube and a line perpendicular to the given plane. Then the lengths of the projections of the cube's edges to this plane take values $a\sin\alpha$, $a\sin\beta$ and $a\sin\gamma$ and each value is taken exactly 4 times. Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ (Problem 1.21), it follows that

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2.$$

Therefore, the desired sum of squares is equal to $8a^2$.

1.25. Through each edge of the tetrahedron draw the plane parallel to the opposite edge. As a result we get a cube into which the given tetrahedron is inscribed; the length of the cube's edge is equal to $\frac{a}{\sqrt{2}}$. The projection of each of the face of the cube is a parallelogram whose diagonals are equal to the projections of the tetrahedron's edges. The sum of squared lengths of the parallelogram's diagonals is equal to the sum of squared lengths of all its edges. Therefore, the sum of squared lengths of two opposite edges of the tetrahedron is equal to the sum of squared lengths of the projections of two pairs of the cube's opposite edges.

Therefore, the sum of squared lengths of the projections of the tetrahedron's edges is equal to the sum of squared lengths of the projections of the cube's edges, i.e., it is equal to $4a^2$.

1.26. As in the preceding problem, let us assume that the vertices of tetrahedron AB_1CD_1 sit in vertices of cube $ABCDA_1B_1C_1D_1$; the length of this cube's edge is equal to $\frac{a}{\sqrt{2}}$. Let O be the center of the tetrahedron. The lengths of segments OA and OD_1 are halves of those of the diagonals of parallelogram ABC_1D_1 and, therefore, the sum of squared lengths of their projections is equal to one fourth of the sum of squared lengths of the projections of this parallelogram's sides.

Similarly, the sum of squared lengths of the projections of segments OC and OB_1 is equal to one fourth of the sum of squared lengths of the projections of the sides of parallelogram A_1B_1CD.

Further, notice that the sum of the squared lengths of the projections of the diagonals of parallelograms AA_1D_1D and BB_1C_1C is equal to the sum of squared lengths of the projections of their edges. As a result we see that the desired sum of squared lengths is equal to one fourth of the sum of squared lengths of the projections of the cube's edges, i.e., it is equal to a^2.

1.27. Let (x_1, y_1, z_1) be the coordinates of the base of the perpendicular dropped from the given point to the given plane. Since vector (a, b, c) is perpendicular to the given plane (Problem 1.7), it follows that $x_1 = x_0 + \lambda a$, $y_1 = y_0 + \lambda b$ and $z_1 = z_0 + \lambda c$, where the distance to be found is equal to $|\lambda|\sqrt{a^2+b^2+c^2}$. Point (x_1, y_1, z_1) lies in the given plane and, therefore,

$$a(x_0 + \lambda a) + (b(y_0 + \lambda b) + c(z_0 + \lambda c) + d = 0,$$

i.e., $\lambda = -\frac{ax_0+by_0+cz_0+d}{a^2+b^2+c^2}$.

1.28. Let us introduce the coordinate system so that the coordinates of points A and B are $(-a, 0, 0)$ and $(a, 0, 0)$, respectively. If the coordinates of point M are (x, y, z), then

$$\frac{AM^2}{BM^2} = \frac{(x+a)^2 + y^2 + z^2}{(x-a)^2 + y^2 + z^2}.$$

The equation $AM : BM = k$ reduces to the form

$$\left(x + \frac{1+k^2}{1-k^2}a\right)^2 + y^2 + z^2 = \left(\frac{2ka}{1-k^2}\right)^2.$$

This equation is an equation of the sphere with center $\left(-\frac{1+k^2}{1-k^2}a, 0, 0\right)$ and radius $\left|\frac{2ka}{1-k^2}\right|$.

1.29. Let us introduce the coordinate system directing the Oz-axis perpendicularly to plane ABC. Let the coordinates of point X be (x, y, z). Then $AX^2 = (x - a_1)^2 + (y - a_2)^2 + z^2$. Therefore, for the coordinates of point X we get an equation of the form

$$(p + q + r)(x^2 + y^2 + z^2) + \alpha x + \beta y + \delta = 0,$$

i.e., $\alpha x + \beta y + \delta = 0$. This equation determines a plane perpendicular to plane ABC. (In particular cases this equation determines the empty set or the whole space.)

1.30. Let the axis of the cone be parallel to the Oz-axis; let the coordinates of the vertex be (a, b, c); α the angle between the axis of the cone and the generator. Then the points from the surface of the cone satisfy the equation

$$(x - a)^2 + (y - b)^2 = k^2(z - c)^2,$$

where $k = \tan\alpha$. The difference of two equations of conic sections with the same angle α is a linear equation; all generic points of conic sections lie in the plane given by this linear equation.

1.31. Let us introduce a coordinate system directing the axes Ox, Oy and Oz along rays AB, AD and AA_1, respectively. Line AA_1 is given by equations $x = 0$, $y = 0$; line CD by equations $y = a$, $z = 0$; line B_1C_1 by equations $x = a$, $z = a$.

Therefore, the squared distances from the point with coordinates (x, y, z) to lines AA_1, CD and B_1C_1 are equal to $x^2 + y^2$, $(y - a)^2 + z^2$ and $(x - a)^2 + (z - a)^2$, respectively. All these numbers cannot be simultaneously smaller than $\frac{1}{2}a^2$ because

$$x^2 + (x - a)^2 \geq \frac{a^2}{2},\ y^2 + (y - a)^2 \geq \frac{a^2}{2} \text{ and } z^2 + (z - a)^2 \geq \frac{1}{2}a^2.$$

All these numbers are equal to $\frac{1}{2}a^2$ for the point with coordinates $\left(\frac{1}{2}a, \frac{1}{2}a, \frac{1}{2}a\right)$, i.e., for the center of the cube.

1.32. Let us direct the coordinate axes Ox, Oy and Oz along rays OA, OB and OC, respectively. Let the angles formed by line l with these axes be equal to α, β and γ, respectively. The coordinates of point M are equal to the coordinates of the projections of points A_1, B_1 and C_1 to axes Ox, Oy and Oz, respectively, i.e., they are equal to $a\cos 2\alpha$, $a\cos 2\beta$ and $a\cos 2\gamma$, where $a = |OA|$. Since

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - 3 = -1$$

(see Problem 1.21) and $-1 \leq \cos 2\alpha, \cos 2\beta, \cos 2\gamma \leq 1$, it follows that the locus to be found consists of the intersection points of the cube determined by conditions $|x|, |y|, |z| \leq a$ with the plane $x+y+z = -a$; this plane passes through the vertices with coordinates $(a, -a, -a)$, $(-a, a, -a)$ and $(-a, -a, a)$.

CHAPTER 2. PROJECTIONS, SECTIONS, UNFOLDINGS

§1. Auxiliary projections

2.1. Given parallelepiped $ABCDA_1B_1C_1D_1$ and the intersection point M of diagonal AC_1 with plane A_1BD. Prove that $AM = \frac{1}{3}AC_1$.

2.2. a) In cube $ABCDA_1B_1C_1D_1$ the common perpendicular MN to lines A_1B and B_1C is drawn so that point M lies on line A_1B. Find the ratio $A_1M : MB$.

b) Given cube $ABCDA_1B_1C_1D_1$ and points M and N on segments AA_1 and BC_1 such that lines MN and B_1D intersect. Find the difference between ratios $BC_1 : BN$ and $AM : AA_1$.

2.3. The angles between a plane and the sides of an equilateral triangle are equal to α, β and γ. Prove that the sine of one of these angles is equal to the sum of sines of the other two angles.

2.4. At the base of the pyramid lies a polygon with an odd number of sides. Is it possible to place arrows on the edges of the pyramid so that the sum of the obtained vectors is equal to zero?

2.5. A plane passing through the midpoints of edges AB and CD of tetrahedron $ABCD$ intersects edges AD and BC at points L and N. Prove that $BC : CN = AD : DL$.

2.6. Given points A, A_1, B, B_1, C, C_1 in space not in one plane and such that vectors $\{AA_1\}$, $\{BB_1\}$ and $\{CC_1\}$ have the same direction. Planes ABC_1, AB_1C and A_1BC intersect at point P and planes A_1B_1C, A_1BC_1 and AB_1C_1 intersect at point P_1. Prove that $PP_1 \parallel AA_1$.

2.7. Given plane Π and points A and B outside it find the locus of points X in plane Π for which lines AX and BX form equal angles with plane Π.

2.8. Prove that the sum of the lengths of edges of a convex polyhedron is greater than $3d$, where d is the greatest distance between the vertices of the polyhedron.

§2. The theorem on three perpendiculars

2.9. Line l is not perpendicular to plane Π, let l' be its projection to plane Π. Let l_1 be a line in plane Π. Prove that $l \perp l_1$ if and only if $l' \perp l_1$. (*Theorem on three perpendiculars.*)

2.10. a) Prove that the opposite edges of a regular tetrahedron are perpendicular to each other.

b) The base of a regular pyramid with vertex S is polygon $A_1 \dots A_{2n-1}$. Prove that edges SA_1 and A_nA_{n+1} are perpendicular to each other.

2.11. Prove that the opposite edges of a tetrahedron are pairwise perpendicular if and only if one of the heights of the tetrahedron passes through the intersection point of the heights of a face (in this case the other heights of the tetrahedron pass through the intersection points of the heights of the faces).

2.12. Edge AD of tetrahedron $ABCD$ is perpendicular to face ABC. Prove that the projection to plane BCD maps the orthocenter of triangle ABC into the orthocenter of triangle BCD.

§3. The area of the projection of a polygon

2.13. The area of a polygon is equal to S. Prove that the area of its projection to plane Π is equal to $S\cos\varphi$, where φ is the angle between plane Π and the plane of the polygon.

2.14. Compute the cosine of the dihedral angle at the edge of a regular tetrahedron.

2.15. The dihedral angle at the base of a regular n-gonal pyramid is equal to α. Find the dihedral angle between its neighbouring lateral faces.

2.16. In a regular truncated quadrilateral pyramid, a section is drawn through the diagonals of the base and another section passing through the side of the lower base. The angle between the sections is equal to α. Find the ratio of the areas of the sections.

2.17. The dihedral angles at the edges of the base of a triangular pyramid are equal to α, β and γ; the areas of the corresponding lateral faces are equal to S_a, S_b and S_c. Prove that the area of the base is equal to

$$S_a\cos\alpha + S_b\cos\beta + S_c\cos\gamma.$$

§4. Problems on projections

2.18. The projections of a spatial figure to two intersecting planes are straight lines. Is this figure necessarily a straight line itself?

2.19. The projections of a body to two planes are disks. Prove that the radii of these disks are equal.

2.20. Prove that the area of the projection of a cube with edge 1 to a plane is equal to the length of its projection to a line perpendicular to this plane.

2.21. Given triangle ABC, prove that there exists an orthogonal projection of an equilateral triangle to a plane so that its projection is similar to the given triangle ABC.

2.22. The projections of two convex bodies to three coordionate planes coincide. Must these bodies have a common point?

§5. Sections

2.23. Given two parallel planes and two spheres in space so that the first sphere is tangent to the first plane at point A and the second sphere is tangent to the second plane at point B and both spheres are tangent to each other at point C. Prove that points A, B and C lie on one line.

2.24. A truncated cone whose bases are great circles of two balls is circumscribed around another ball (cf. Problem 4.18). Determine the total area of the cone's surface if the sum of surfaces of the three balls is equal to S.

2.25. Two opposite edges of a tetrahedron are perpendicular and their lengths are equal to a and b; the distance between them is equal to c. A cube four edges of which are perpendicular to these two edges of the tetrahedron is inscribed in the tetrahedron and on every face of the tetrahedron exactly two vertices of the cube lie. Find the length of the cube's edge.

2.26. What regular polygons can be obtained when a plane intersects a cube?

2.27. All sections of a body by planes are disks. Prove that this body is a ball.

2.28. Through vertex A of a right circular cone a section of maximal area is drawn. The area of this section is twice that of the section passing through the axis of the cone. Find the angle at the vertex of the axial section of the cone.

2.29. A plane divides the medians of faces ABC, ACD and ADB of tetrahedron $ABCD$ originating from vertex A in ratios of $2:1$, $1:2$ and $4:1$ counting from vertex A. Let P, Q and R be the intersection points of this plane with lines AB, AC and AD. Find ratios $AP : PB$, $AQ : QS$ and $AR : RD$.

2.30. In a regular hexagonal pyramid $SABCDEF$ (with vertex S) three points are taken on the diagonal AD that divide it into 4 equal parts. Through these points sections parallel to plane SAB are drawn. Find the ratio of areas of the obtained sections.

2.31. A section of a regular quadrilateral pyramid is a regular pentagon. Prove that the lateral faces of this pyramid are equilateral triangles.

§6. Unfoldings

2.32. Prove that all the faces of tetrahedron $ABCD$ are equal if and only if one of the following conditions holds:

a) sums of the plane angles at some three vertices of the tetrahedron are equal to 180°;

b) sums of the plane angles at some two vertices are equal to 180° and, moreover, some two opposite edges are equal;

c) the sum of the plane angles at some vertex is equal to 180° and, moreover, there are two pairs of equal opposite edges in the tetrahedron.

2.33. Prove that if the sum of the plane angles at a vertex of a pyramid is greater than 180°, then each of its lateral edges is smaller than a semiperimeter of the base.

2.34. Let S_A, S_B, S_C and S_D be the sums of the plane angles of tetrahedron $ABCD$ at vertices A, B, C and D, respectively. Prove that if $S_A = S_B$ and $S_C = S_D$, then $\angle ABC = \angle BAD$ and $\angle ACD = \angle BDC$.

Problems for independent study

2.35. The length of the edge of cube $ABCDA_1B_1C_1D_1$ is equal to a. Let P, K and L be the midpoints of edges AA_1, A_1D_1 and B_1C_1; let Q be the center of face CC_1D_1D. Segment MN with the endpoints on lines AD and KL intersects line PQ and is perpendicular to it. Find the length of this segment.

2.36. The number of vertices of a polygon is equal to n. Prove that there is a projection of this polygon the number of vertices of which is a) not less than 4; b) not greater than $n-1$.

2.37. Projections of a right triangle to faces of a dihedral angle of value α are equilateral triangles with side 1 each. Find the hypothenuse of the right triangle.

2.38. Prove that if the lateral surface of a cylinder is intersected by a slanted plane and then cut along the generator and unfolded onto a plane, then the curve of the section is a graph oof the sine function.

2.39. The volume of tetrahedron $ABCD$ is equal to 5. Through the midpoints of edges AD and BC a plane is drawn that intersects edge CD at point M and $DM : CM = 2 : 3$. Compute the area of the section of the tetrahedron with the indicated plane if the distance from vertex A to the plane is equal to 1.

2.40. In a regular quadrilateral pyramid $SABCD$ with vertex S, a side at the base is equal to a and the angle between a lateral edge and the plane of the base is equal to α. A plane parallel to AC and BS intersects pyramid so that a circle can be inscribed in the section. Find the radius of this circle.

2.41. The length of an edge of a regular tetrahedron is equal to a. Plane Π passes through vertex B and the midpoints of edges AC and AD. A ball is tangent to lines AB, AC, AD and the part of plane Π, which is confined inside the tetrahedron. Find the radius of this ball.

2.42. The edge of a regular tetrahedron $ABCD$ is equal to a. Let M be the center of face ADC; let N be the midpoint of edge BC. Find the radius of the ball inscribed in the trihedral angle A and tangent to line MN.

2.43. The dihedral angle at edge AB of tetrahedron $ABCD$ is a right one; M is the midpoint of edge CD. Prove that the area of triangle AMB is four times smaller than the area of the parallelogram whose sides are equal and parallel to segments AB and CD.

Solutions

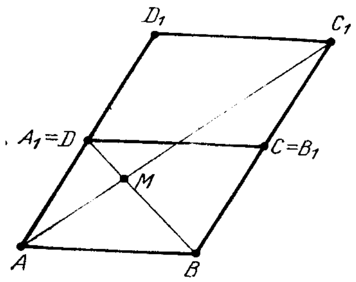

Figure 17 (Sol. 2.1)

2.1. Consider the projection of the given parallelepiped to plane ABC parallel to line A_1D (Fig. 17). From this figure it is clear that

$$AM : MC_1 = AD : BC_1 = 1 : 2.$$

2.2. a) **First solution.** Consider projection of the given cube to a plane perpendicular to line B_1C (Fig. 18 a)). On this figure, line B_1C is depicted by a dot and segment MN by the perpendicular dropped from this dot to line A_1B. It is also clear that, on the figure, $A_1B_1 : B_1B = \sqrt{2} : 1$. Since $A_1M : MN = A_1B_1 : B_1B$ and $MN : MB = A_1B_1 : B_1B$, it follows that $A_1M : MB = A_1B_1^2 : B_1B^2 = 2 : 1$.

Second solution. Consider the projection of the given cube to the plane perpendicular to line AC_1 (Fig. 18 b). Line AC_1 is perpendicular to the planes of triangles A_1BD and B_1CD_1 and, therefore, it is perpendicular to lines A_1B and B_1C, i.e., segment MN is parallel to AC_1. Thus, segment MN is plotted on the projection by the dot — the intersection point of segments A_1B and B_1C. Therefore, on segment MN we have

$$A_1M : MB = A_1C : BB_1 = 2 : 1.$$

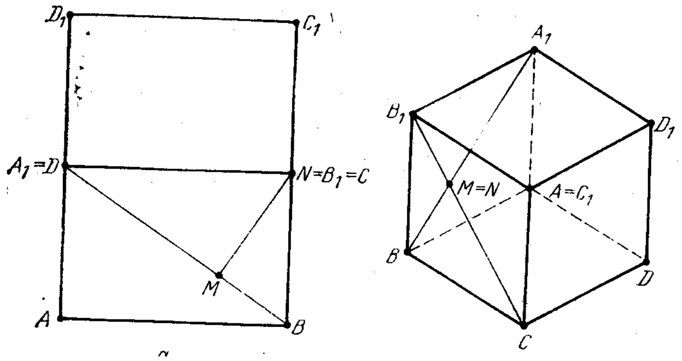

FIGURE 18 (SOL. 2.2 A))

b) Consider the projection of the cube to the plane perpendicular to diagonal B_1D (Fig. 19). On the projection, hexagon $ABCC_1D_1A_1$ is a regular one and line MN passes through its center; let L be the intersection point of lines MN and AD_1, P the intersection point of line AA_1 with the line passing through point D_1 parallel to MN. It is easy to see that $\triangle ADM = \triangle A_1D_1P$; hence, $AM = A_1P$. Therefore,

$$BC_1 : BN = AD_1 : D_1L = AP : PM = (AA_1 + AM) : AA_1 = 1 + AM : AA_1,$$

i.e., the desired difference of ratios is equal to 1.

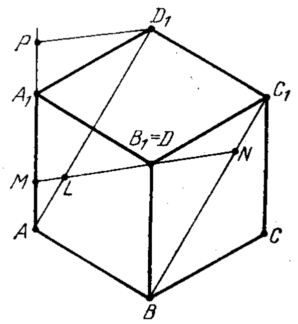

FIGURE 19 (SOL. 2.2 B))

2.3. Let A_1, B_1 and C_1 be the projections of the vertices of the given equilateral triangle ABC to a line perpendicular to the given plane. If the angles between the given plane and lines AB, BC and CA are equal to γ, α and β, respectively, then $A_1B_1 = a\sin\gamma$, $B_1C_1 = a\sin\alpha$ and $C_1A_1 = a\sin\beta$, where a is the length of the side of triangle ABC. Let, for definiteness sake, point C_1 lie on segment A_1B_1. Then $A_1B_1 = A_1C_1 + C_1B_1$, i.e., $\sin\gamma = \sin\alpha + \sin\beta$.

2.4. No, this is impossible. Consider the projection to a line perpendicular to the base. The projections of all the vectors from the base are zeros and the projection of the sum of vectors of the lateral edges cannot be equal to zero since the sum of an odd number of 1's and -1's is odd.

2.5. Consider the projection of the tetrahedron to a plane perpendicular to the line that connects the midpoints of edges AB and CD. This projection maps the given plane to line LN that passes through the intersection point of the diagonals of parallelogram $ADBC$. Clearly, the projections satisfy

$$B'C' : C'N' = A'D' : D'L'.$$

2.6. Let K be the intersection point of segments BC_1 and B_1C. Then planes ABC_1 and AB_1C intersect along line AK and planes A_1B_1C and A_1BC_1 intersect along line A_1K. Consider the projection to plane ABC parallel to AA_1. Both the projection of point P and the projection of point P_1 lie on line AK_1, where K_1 is the projection of point K.

Similar arguments show that the projections of points P and P_1 lie on lines BL_1 and CM_1, respectively, where L_1 is the projection of the intersection point of lines AC_1 and A_1C, M_1 is the projection of the intersection point of lines AB_1 and A_1B. Therefore, the projections of points P and P_1 coincide, i.e., $PP_1 \parallel AA_1$.

2.7. Let A_1 and B_1 be the projections of points A and B to plane Π. Lines AX and BX form equal angles with plane Π if and only if the right triangles AA_1X and BB_1X are similar, i.e., $A_1X : B_1X = A_1A : B_1B$. The locus of the points in plane the ratio of whose distances from two given points A_1 and B_1 of the same plane is either an *Apollonius's circle* or a line, see Plain 13.7).

2.8. Let $d = AB$, where A and B are vertices of the polyhedron. Consider the projection of the polyhedron to line AB. If the projection of point C lies not on segment AB but on its continuation, say, beyond point B, then $AC > AB$.

Therefore, all the points of the polyhedron are mapped into points of segment AB. Since the length of the projection of a segment to a line does not exceed the length of the segment itself, it suffices to show that the projection maps points of at least theree distinct edges into every inner point of segment AB. Let us draw a plane perpendicular to segment AB through an arbitrary inner point of AB. The section of the polyhedron by this plane is an n-gon, where $n \geq 3$, and, therefore, the plane intersects at least three distinct edges.

2.9. Let O be the intersection point of line l and plane Π (the case when line l is parallel to plane Π is obvious); A an arbitrary point on line l distinct from O; A' its projection to plane Π. Line AA' is perpendicular to any line in plane Π; hence, $AA' \perp l_1$. If $l \perp l_1$, then $AO \perp l_1$; hence, line l_1 is perpendicular to plane AOA' and, therefore, $A'O \perp l_1$. If $l' \perp l_1$, then the considerations are similar.

2.10. Let us solve heading b) whose particular case is heading a). The projection of vertex S to the plane at the base is the center O of a regular polygon $A_1 \dots A_{2n-1}$ and the projection of line SA_1 to this plane is line OA_1. Since $OA_1 \perp A_nA_{n+1}$, it follows that $SA_1 \perp A_nA_{n+1}$, cf. Problem 2.9.

2.11. Let AH be a height of tetrahedron $ABCD$. By theorem on three perpendiculars $BH \perp CD$ if and only if $AB \perp CD$.

2.12. Let BK and BM be heights of triangles ABC and DBC, respectively. Since $BK \perp AC$ and $BK \perp AD$, line BK is perpendicular to plane ADC and, therefore, $BK \perp DC$. By the theorem on three perpendiculars the projection of line BK to plane BDC is perpendicular to line DC, i.e., the projection coincides with line BM.

For heights dropped from vertex C the proof is similar.

2.13. The statement of the problem is obvious for the triangle one of whose sides is parallel to the intersection line of plane Π with the plane of the polygon.

Indeed the length of this side does not vary under the projection and the length of the height dropped to it changes under the projection by a factor of $\cos\varphi$.

Now, let us prove that any polygon can be cut into the triangles of the indicated form. To this end let us draw through all the vertices of the polygon lines parallel to the intersection line of the planes. These lines divide the polygon into triangles and trapezoids. It remains to cut each of the trapezoids along any of its diagonals.

2.14. Let φ be the dihedral angle at the edge of the regular tetrahedron; O the projection of vertex D of the regular tetrahedron $ABCD$ to the opposite face. Then

$$\cos\varphi = S_{ABO} : S_{ABD} = \frac{1}{3}.$$

2.15. Let S be the area of the lateral face, h the height of the pyramid, a the length of the side at the base and φ the angle to be found. The area of the projection to the bisector plane of the dihedral angle between the neighbouring lateral faces is equal for each of these faces to $S\cos\frac{\varphi}{2}$; on the other hand, it is equal to $\frac{1}{2}ah\sin\frac{\pi}{n}$.

It is also clear that the area of the projection of the lateral face to the plane passing through its base perpendicularly to the base of the pyramid is equal to $S\sin\alpha$; on the other hand, it is equal to $\frac{1}{2}ah$. Therefore,

$$\cos\frac{\varphi}{2} = \sin\alpha\sin\frac{\pi}{n}.$$

2.16. The projection of a side of the base to the plane of the first section is a half of the diagonal of the base and, therefore, the area of the projection of the second section to the plane of the first section is equal to a half area of the first section. On the other hand, if the area of the second section is equal to S, then the area of its projection is equal to $S\cos\alpha$ and, therefore, the area of the first section is equal to $2S\cos\alpha$.

2.17. Let D' be the projection of vertex D of pyramid $ABCD$ to the plane of the base. Then

$$S_{ABC} = \pm S_{BCD'} \pm S_{ACD'} \pm S_{ABD'} = S_a\cos\alpha + S_b\cos\beta + S_c\cos\gamma.$$

The area of triangle BCD' is taken with a "$-$" sign if points D' and A lie on distinct sides of line BC and with a $+$ sign otherwise; for areas of triangles ACD' and ABD' the sign is similarly selected.

2.18. Not necessarily. Consider a plane perpendicular to the two given planes. Any figure in this plane possesses the required property only if the projections of the figure on the given planes are unbounded.

2.19. The diameters of the indicated disks are equal to the length of the projection of the body to the line along which the given planes intersect.

2.20. Let the considered projection send points B_1 and D into inner points of the projection of the cube (Fig. 20). Then the area of the projection of the cube is equal to the doubled area of the projection of triangle ACD_1, i.e., it is equal to $2S\cos\varphi$, where S is the area of triangle ACD_1 and φ is the angle between the plane of the projection and plane ACD_1. Since the side of triangle ACD_1 is equal to $\sqrt{2}$, we deduce that $2S = \sqrt{3}$.

The projection of the cube to line l perpendicular to the plane of the projection coincides with the projection of diagonal B_1D to l. Since line B_1D is perpendicular

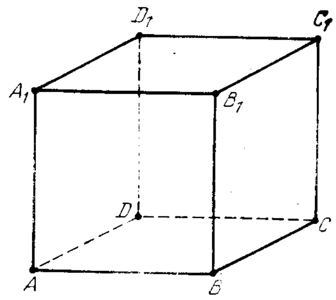

FIGURE 20 (SOL. 2.20)

to plane ACD_1, the angle between lines l and B_1D is also equal to φ. Therefore, the length of the projection of the cube to line l is equal to

$$B_1D\cos\varphi = \sqrt{3}\cos\varphi.$$

2.21. Let us draw lines perpendicular to plane ABC through vertices A and B and select points A_1 and B_1 on them. Let $AA_1 = x$ and $BB_1 = y$ (if points A_1 and B_1 lie on different sides of plane ABC, then we assume that the signs of x and y are distinct). Let a, b and c be the lengths of the sides of the given triangle. It suffices to verify that numbers x and y can be selected so that triangle A_1B_1C is an equilateral one, i.e., so that

$$x^2 + b^2 = y^2 + a^2 \text{ and } (x^2 - y^2)^2 + c^2 = y^2 + a^2.$$

Let

$$a^2 - b^2 = \lambda \text{ and } a^2 - c^2 = \mu, \text{ i.e., } x^2 - y^2 = \lambda \text{ and } x^2 - 2xy = \mu.$$

From the second equation we deduce that $2y = x - \frac{\mu}{x}$. Inserting this expression into the first equation we get equation

$$3u^2 + (2\mu - 4\lambda)u - \mu^2 = 0, \text{ where } u = x^2.$$

The discriminant D of this quadratic equation is non-negative and, therefore, the equation has a root x. If $x \neq 0$, then $2y = x - \frac{\mu}{x}$. It remains to notice that if $x = 0$ is the only solution of the obtained equation, i.e., $D = 0$, then $\lambda = \mu = 0$ and, therefore, $y = 0$ is a solution.

2.22. They must. First, let us prove that if the projections of two convex planar figures to the coordinate axes coincide, then these figures have a common point. To this end it suffices to prove that if points K, L, M and N lie on sides AB, BC, CD and DA of rectangle $ABCD$, then the intersection point of diagonals AC and BD belongs to quadrilateral $KLMN$.

Diagonal AC does not belong to triangles KBL and NDM and diagonal BD does not belong to triangulars KAN and LCM. Therefore, the intersection point of diagonals AC and BD does not belong to either of these triangles; hence, it belongs to quadrilateral $KLMN$.

The base planes parallel to coordinate ones coincide for the bodies considered. Let us take one of the base planes. The points of each of the considered bodies that lie in this plane constitute a convex figure and the projections of these figures to the coordinate axes coincide. Therefore, in each base plane there is at least one common point of the considered bodies.

2.23. Points A, B and C lie in one plane in any case, consequently, we can consider the section by the plane that contains these points. Since the plane of the section passes through the tangent points of spheres (of the sphere and the plane), it follows that in the section we get tangent circles (or a line tangent to a circle). Let O_1 and O_2 be the centers of the first and second circles. Since $O_1A \parallel O_2B$ and points O_1, C and O_2 lie on one line, we have $\angle AO_1C = \angle BO_2C$. Hence, $\angle ACO_1 = \angle BCO_2$, i.e., points A, B and C lie on one line.

2.24. The axial section of the given truncated cone is the circumscribed trapezoid $ABCD$ with bases $AD = 2R$ and $BC = 2r$. Let P be the tangent point of the inscribed circle with side AB, let O be the center of the inscribed circle. In triangle ABO, the sum of the angles at vertices A and B is equal to 90° because $\triangle ABO$ is a right one. Therefore, $AP : PO = PO : BP$, i.e., $PO^2 = AP \cdot BP$. It is also clear that $AP = R$ and $BP = r$. Therefore, the radius PO of the sphere inscribed in the cone is equal to $\sqrt{Rr}$; hence,

$$S = 4\pi(R^2 + Rr + r^2).$$

Expressing the volume of the given truncated cone with the help of the formulas given in the solutions of Problems 3.7 and 3.11 and equating these expressions we see that the total area of the cone's surface is equal to

$$2\pi(R^2 + Rr + r^2) = \frac{S}{2}$$

(take into account that the height of the truncated cone is equal to the doubled radius of the sphere around which it is circumscribed).

2.25. The common perpendicular to the given edges is divided by the planes of the cube's faces parallel to them into segments of length y, x and z, where x is the length of the cube's edge and y is the length of the segment adjacent to edge a. The planes of the cube's faces parallel to the given edges intersect the tetrahedron along two rectangles. The shorter sides of these rectangles are of the same length as that of the cube's edge, x. The sides of these rectangles are easy to compute and we get $x = \frac{by}{c}$ and $x = \frac{az}{c}$. Therefore,

$$c = x + y + z = x + \frac{cx}{b} + \frac{cx}{a}, \text{ i.e., } x = \frac{abc}{ab + bc + ca}.$$

2.26. Each side of the obtained polygon belongs to one of the faces of the cube and, therefore, the number of its sides does not exceed 6. Moreover, the sides that belong to the opposite faces of the cube are parallel, because the intersection lines of the plane with two parallel planes are parallel. Hence, the section of the cube cannot be a regular pentagon: indeed, such a pentagon has no parallel sides. It is easy to verify that an equilateral triangle, square, or a regular hexagon can be sections of the cube.

2.27. Consider the disk which is a section of the given body. Let us draw through its center line l perpendicular to its plane. This line intersects the given body along segment AB. All the sections passing through line l are disks with diameter AB.

2.28. Consider an arbitrary section passing through vertex A. This section is triangle ABC and its sides AB and AC are generators of the cone, i.e., have a constant length. Hence, the area of the section is proportional to $\sin BAC$. Angle BAC varies from $0°$ to φ, where φ is the angle at the vertex of the axial section of the cone. If $\varphi \leq 90°$, then the axial section is of the maximal area and if $\varphi > 90°$, then the section with the right angle at vertex A is of maximal area. Therefore, the conditions of the problem imply that $\sin\varphi = 0.5$ and $\varphi > 90°$, i.e., $\varphi = 120°$.

2.29. Let us first solve the following problem. Let on sides AB and AC of triangle ABC points L and K be taken so that $AL : LB = m$ and $AK : KC = n$; let N be the intersection point of line KL and median AM. Let us compute the ratio $AN : NM$.

To this end consider points S and T at which line KL intersects line BC and the line drawn through point A parallel to BC, respectively. Clearly, $AT : SB = AL : LB = m$ and $AT : SC = AK : KC = n$. Hence,

$$AN : NM = AT : SM = 2AT : (SC + SB) = 2(SC : AT + SB : AT)^{-1} = \frac{2mn}{m+n}.$$

Observe that all the arguments remain true in the case when points K and L are taken on the continuations of the sides of the triangle; in which case the numbers m and n are negative.

Now, suppose that $AP : PB = p$, $AQ : QC = q$ and $AR : RD = r$. Then by the hypothesis

$$\frac{2pq}{p+q} = 2, \quad \frac{2qr}{q+r} = \frac{1}{2}, \quad \text{and} \quad \frac{2pr}{p+r} = 4.$$

Solving this system of equations we get $p = -\frac{4}{5}$, $q = \frac{4}{9}$ and $r = \frac{4}{7}$. The minus sign of p means that the given plane intersects not the segment AB but its continuation.

2.30. Let us number the given sections (planes) so that the first of them is the closest to vertex A and the third one is the most distant from A. Considering the projection to the plane perpendicular to line CF it is easy to see that the first plane passes through the midpoint of edge SC and divides edge SD in the ratio of 1:3 counting from point S; the second plane passes through the midpoint of edge SD and the third one divides it in the ratio of 3:1.

Let the side of the base of the pyramid be equal to $4a$ and the height of the lateral face be equal to $4h$. Then the first section consists of two trapezoids: one with height $2h$ and bases $6a$ and $4a$ and the other one with height h and bases $4a$ and a. The second section is a trapezoid with height $2h$ and bases $8a$ and $2a$. The third section is a trapezoid with height h and bases $6a$ and $3a$. Therefore, the ratio of areas of the sections is equal to 25:20:9.

2.31. Since a quadrilateral pyramid has five faces, the given section passes through all the faces. Therefore, we may assume that vertices K, L, M, N and O of the regular pentagon lie on edges AB, BC, CS, DS and AS, respectively. Consider the projection to the plane perpendicular to edge BC (Fig. 21). Let $B'K' : A'B' = p$. Since $M'K' \parallel N'O'$, $M'O' \parallel K'L'$ and $K'N' \parallel M'L'$, it follows that

$$B'M' : B'S' = A'O' : A'S' = S'N' : A'S' = p.$$

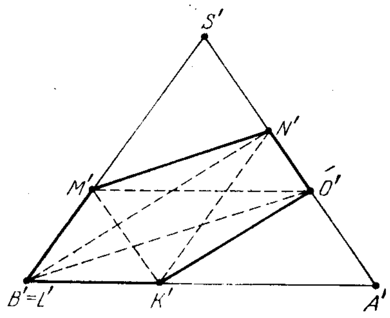

FIGURE 21 (SOL. 2.31)

Therefore, $S'O' : A'S' = 1-p$; hence, $S'N' : A'S' = (1-p)^2$ because $M'N' \parallel L'O'$. Thus, $p = S'N' : A'S' = (1-p)^2$, i.e., $p = \frac{3-\sqrt{5}}{2}$.

Let $SA = 1$ and $\angle ASB = 2\varphi$. Then

$$NO^2 = p^2 + (1-p)^2 - 2p(1-p)\cos 2\varphi$$

and

$$KO^2 = p^2 + 4(1-p)^2 \sin^2\varphi - 4p(1-p)\sin^2\varphi.$$

Equating these expressions and taking into account that $\cos 2\varphi = 1 - 2\sin^2\varphi$ let us divide the result by $1-p$. We get

$$1 - 3p = 4(1-3p)\sin^2\varphi.$$

Since in our case $1 - 3p \neq 0$, it follows that $\sin^2\varphi = \frac{1}{4}$, i.e., $\varphi = 30^\circ$.

2.32. a) Let the sum of the plane angles at vertices A, B and C be equal to 180°. Then the unfolding of the tetrahedron to plane ABC is a triangle and points A, B and C are the midpoints of the triangle's sides. Hence, all the faces of the tetrahedron are equal.

Conversely, if all the faces of the tetrahedron are equal, then any two neighbouring faces constitute a parallelogram in its unfolding. Hence, the unfolding of the tetrahedron is a triangle, i.e., the sums of plane angles at the vertices of the tetrahedron are equal to 180°.

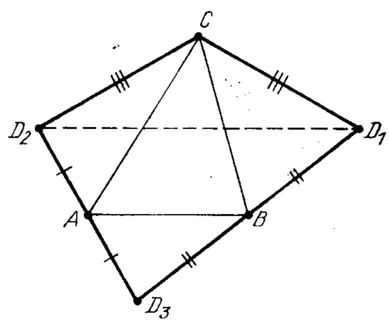

FIGURE 22 (SOL. 2.32)

b) Let the sums of plane angles at vertices A and B be equal to $180°$. Let us consider the unfolding of the tetrahedron to the plane of face ABC (Fig. 22). Two variants are possible.

1) Edges AB and CD are equal. Then

$$D_1C + D_2C = 2AB = D_1D_2;$$

hence, C is the midpoint of segment D_1D_2.

2) Edges distinct from AB and CD are equal. Let, for definiteness, $AC = BD$. Then point C belongs to both the midperpendicular to segment D_1D_2 and to the circle of radius BD centered at A. One of the intersection points of these sets is the midpoint of segment D_1D_2 and the other intersection point lies on the line passing through D_3 parallel to D_1D_2. In our case the second point does not fit.

c) Let the sum of plane angles at vertex A be equal to $180°$, $AB = CD$ and $AD = BC$. Let us consider the unfolding of the tetrahedron to plane ABC and denote the images of vertex D as plotted on Fig. 22. The opposite sides of quadrilateral $ABCD_2$ are equal, hence, it is a parallelogram. Therefore, segments CB and AD_3 are parallel and equal and, therefore, $ACBD_3$ is a parallelogram. Thus, the unfolding of the tetrahedron is a triangle and A, B and C are the midpoints of its sides.

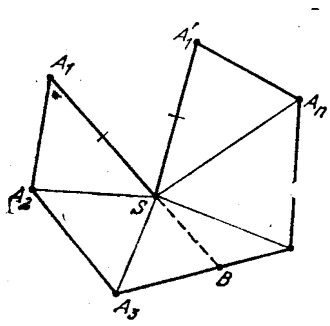

FIGURE 23 (SOL. 2.33)

2.33. Let $SA_1 \dots A_n$ be the given pyramid. Let us cut its lateral surface along edge SA_1 and unfold it on the plane (Fig. 23). By the hypothesis point S lies inside polygon $A_1 \dots A_n A_1'$. Let B be the intersection point of the extension of segment A_1S beyond point S with a side of this polygon. If a and b aree the lengths of broken lines $A_1A_2 \dots B$ and $B \dots A_nA_1'$, then $A_1S + SB < a$ and $A_1'S < SB + b$. Hence, $2A_1S < a + b$.

2.34. Since the sum of the angles of each of the tetrahedron's faces is equal to $180°$, it follows that

$$S_A + S_B + S_C + S_D = 4 \cdot 180°.$$

Let, for definiteness sake, $S_A \leq S_C$. Then $360° - S_C = S_A \leq 180°$. Consider the unfolding of the given tetrahedron to plane ABC (Fig. 24).

Since $\angle AD_3C = \angle D_1D_3D_2$ and $AD_3 : D_3C = D_1D_3 : D_3D_2$, it follows that $\triangle ACD_3 \sim \triangle D_1D_2D_3$ and the similarity coefficient is equal to the ratio of the lateral side to the base in the isosceles triangle with angle S_A at the vertex. Hence, $AC = D_1B$. Similarly, $CB = AD_1$. Therefore, $\triangle ABC = \triangle BAD_1 = \triangle BAD$. We similarly prove that $\triangle ACD = \triangle BDC$.

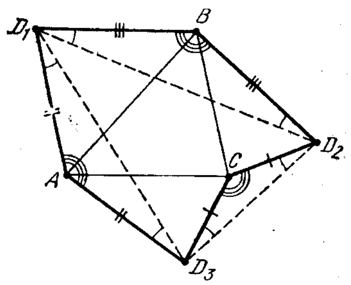

Figure 24 (Sol. 2.34)

CHAPTER 3. VOLUME

§1. Formulas for the volumes of a tetrahedron and a pyramid

3.1. Three lines intersect at point A. On each of them two points are taken: B and B', C and C', D and D', respectively. Prove that

$$V_{ABCD} : V_{AB'C'D'} = (AB \cdot AC \cdot AD) : (AB' \cdot AC' \cdot AD').$$

3.2. Prove that the volume of tetrahedron $ABCD$ is equal to

$$AB \cdot AC \cdot AD \cdot \sin\beta \sin\gamma \sin\frac{\angle D}{6},$$

where β and γ are plane angles at vertex A opposite to edges AB and AC, respectively, and $\angle D$ is the dihedral angle at edge AD.

3.3. The areas of two faces of tetrahedron are equal to S_1 and S_2, a is the length of the common edge of these faces, α the dihedral angle between them. Prove that the volume V of the tetrahedron is equal to $2S_1S_2\sin\frac{\alpha}{3a}$.

3.4. Prove that the volume of tetrahedron $ABCD$ is equal to $dAB \cdot CD\sin\frac{\varphi}{6}$, where d is the distance between lines AB and CD and φ is the angle between them.

3.5. Point K belongs to the base of pyramid of vertex O. Prove that the volume of the pyramid is equal to $S \cdot \frac{KO}{3}$, where S is the area of the projection of the base to the plane perpendicular to KO.

3.6. In parallelepiped $ABCDA_1B_1C_1D_1$, diagonal AC_1 is equal to d. Prove that there exists a triangle the lengths of whose sides are equal to distances from vertices A_1, B and D to diagonal AC_1 and the volume of this parallelepiped is equal to $2dS$, where S is the area of this triangle.

§2. Formulas for the volumes of polyhedrons and bodies of revolution

3.7. Prove that the volume of the polyhedron circumscribed about a sphere of radius R is equal to $\frac{1}{3}$, where S is the area of the polyhedron's surface.

3.8. Prove that the ratio of volumes of the sphere to that of the truncated cone circumscribed about it is equal to the ratio of the total areas of their surfaces.

3.9. A ball of radius R is tangent to one of the bases of a truncated cone and is tangent to its lateral surface along a circle which is the circle of the other base of the cone. Find the volume of the body consisting of the cone and the ball if the total area of the surface of this body is equal to S.

3.10. a) The radius of a right circular cylinder and its height are equal to R. Consider the ball of radius R centered at the center O of the lower base of the cylinder and the cone with vertex at O whose base is the upper base of the cylinder. Prove that the volume of the cone is equal to the volume of the part of the cylinder which lies outside the ball. In the proof make use of the equality of the areas of sections parallel to the bases. (*Archimedus*)

b) Assuming that the formulas for the volume of the cylinder and the cone are known, deduce the formula for the volume of a ball.

3.11. Find the volume V of a truncated cone with height h and with the radii of the bases R and r.

3.12. Given a plane convex figure of perimeter $2p$ and area S. Consider a body consisting of points whose distance from this figure does not exceed d. Find the volume of this body.

3.13. The volume of a convex polygon is equal to V and the area of its surface is equal to S; the length of the i-th edge is equal to l_i, the dihedral angle at this edge is equal to φ_i. Consider the body the distance of whose points to the polygon does not exceed d. Find the volume and the surface area of this body.

3.14. All the vertices of a convex polyhedron lie on two parallel planes. Prove that the volume of the polyhedron is equal to $\frac{1}{6}h(S_1 + S_2 + 4S)$, where S_1 and S_2 are the areas of the faces lying on the given planes and S is the area of the section of the polyhedron by the plane equidistant from the given ones, h the distance between the given plane.

§3. Properties of the volume

3.15. Two skew lines in space are given. The opposite edges of a tetrahedron are moving, as solid bodies, along these lines, whereas the other dimensions of the tetrahedron may vary. Prove that the volume of the tetrahedron does not vary.

3.16. Three parallel lines a, b and c in space are given. An edge of a tetrahedron is moved along line a so that its length does not vary and the two other vertices move along lines b and c. Prove that the volume of tetrahedron does not vary.

3.17. Prove that the plane that only intersects a lateral surface of the cylinder divides its volume in the same ratio in which it divides the axis of the cylinder.

3.18. Prove that a plane passing through the midpoints of two skew edges of a tetrahedron divides it into two parts of equal volume.

3.19. Parallel lines a, b, c and d intersect a plane at points A, B, C and D and another plane at points A', B', C' and D'. Prove that the volumes of tetrahedrons $A'BCD$ and $AB'C'D'$ are equal.

3.20. In the planes of the faces of tetrahedron $ABCD$ points A_1, B_1, C_1 and D_1 are taken so that the lines AA_1, BB_1, CC_1 and DD_1 are parallel. Find the ratio of volumes of tetrahedrons $ABCD$ and $A_1B_1C_1D_1$.

§4. Computation of volumes

3.21. Planes ABC_1 and A_1B_1C divide triangular prism $ABCAB_1C_1$ into four parts. Find the ratio of volumes of these parts.

3.22. The volume of parallelepiped $ABCDA_1B_1C_1D_1$ is equal to V. Find the volume of the common part of tetrahedrons $ABCD_1$ and A_1BC_1D.

3.23. Consider a tetrahedron. A plane is parallel to two of the tetrahedron's skew edges and divides one of the other edges in the ratio of 2:1 What is the ratio in which the volume of a tetrahedron is divided by the plane?

3.24. On two parallel lines we take similarly directed vectors $\{AA_1\}$, $\{BB_1\}$ and $\{CC_1\}$. Prove that the volume of the convex polyhedron $ABCA_1B_1C_1$ is equal to $\frac{1}{3}S(AA_1 + BB_1 + CC_1)$, where S is the area of the triangle obtained at the intersection of these lines by a plane perpendicular to them.

3.25. Let M be the intersection point of the medians of tetrahedron $ABCD$ (see $). Prove that there exists a quadrilateral whose sides are equal to segments that connect M with the vertices of the tetrahedron and are parallel to them. Compute the volume of the tetrahedron given by this spatial quadrilateral if the volume of tetrahedron $ABCD$ is equal to V.

3.26. Through a height of a equilateral triangle with side a a plane perpendicular to the triangle's plane is drawn; in the new plane line l parallel to the height of the triangle is taken. Find the volume of the body obtained after rotation of the triangle about line l.

3.27. Lines AC and BD the angle between which is equal to α ($\alpha < 90^\circ$) are tangent to a ball of radius R at diametrically opposite points A and B. Line CD is also tangent to the ball and the angle between AB and CD is equal to φ ($\varphi < 90^\circ$). Find the volume of tetrahedron $ABCD$.

3.28. Point O lies on the segment that connects the vertex of the triangular pyramid of volume V with the intersection point of medians of the base. Find the volume of the common part of the given pyramid and the pyramid symmetric to it through point O if point O divides the above described segment in the ratio of: a) 1:1; b) 3:1; c) 2:1; d) 4:1 (counting from the vertex).

3.29. The sides of a spatial quadrilateral $KLMN$ are perpendicular to the faces of tetrahedron $ABCD$ and their lengths are equal to the areas of the corresponding faces. Find the volume of tetrahedron $KLMN$ if the volume of tetrahedron $ABCD$ is equal to V.

3.30. A lateral edge of a regular prism $ABCA_1B_1C_1$ is equal to a; the height of the basis of the prism is also equal to a. Planes perpendicular to lines AB and AC_1 are drawn through point A and planes perpendicular to A_1B and A_1C are drawn through point A_1. Find the volume of the figure bounded by these four planes and plane B_1BCC_1.

3.31. Tetrahedrons $ABCD$ and $A_1B_1C_1D_1$ are placed so that the vertices of each of them lie in the corresponding planes of the faces of the other tetrahedron (i.e., A lies in plane $B_1C_1D_1$, etc.). Moreover, A_1 coincides with the intersection point of the medians of triangle BCD and lines BD_1, CB_1 and DC_1 divide segments AC, AD and AB, respectively, in halves. Find the volume of the common part of the tetrahedrons if the volume of tetrahedron $ABCD$ is equal to V.

§5. An auxiliary volume

3.32. Prove that the bisector plane of a dihedral angle at an edge of a tetrahedron divides the opposite edge into parts proportional to areas of the faces that confine this angle.

3.33. In tetrahedron $ABCD$ the areas of faces ABC and ABD are equal to p and q and the angle between them is equal to α. Find the area of the section passing through edge AB and the center of the ball inscribed in the tetrahedron.

3.34. Prove that if x_1, x_2, x_3, x_4 are distances from an arbitrary point inside a tetrahedron to its faces and h_1, h_2, h_3, h_4 are the corresponding heights of the tetrahedron, then

$$\frac{x_1}{h_1}+\frac{x_2}{h_2}+\frac{x_3}{h_3}+\frac{x_4}{h_4}=1.$$

3.35. On face ABC of tetrahedron $ABCD$ a point O is taken and segments OA, OB_1 and OC_1 are drawn through it so that they are parallel to the edges DA, DB and DC, respectively, to the intersection with faces of the tetrahedron. Prove that

$$\frac{OA_1}{DA}+\frac{OB_1}{DB}+\frac{OC_1}{DC}=1.$$

3.36. Let r be the radius of the sphere inscribed in a tetrahedron; r_a, r_b, r_c and r_d the radii of spheres each of which is tangent to one face and the extensions of the other three faces of the tetrahedron. Prove that

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}+\frac{1}{r_d}=\frac{2}{r}.$$

3.37. Given a convex quadrangular pyramid $MABCD$ with vertex M and a plane that intersects edges MA, MB, MC and MD at points A_1, B_1, C_1 and D_1, respectively. Prove that

$$S_{BCD}\frac{MA}{MA_1}+S_{ABD}\frac{MC}{MC_1}=S_{ABC}\frac{MD}{MD_1}+S_{ACD}\frac{MB}{MB_1}.$$

3.38. The lateral faces of a triangular pyramid are of equal area and the angles they constitute with the base are equal to α, β and γ. Find the ratio of the radius of the ball inscribed in this pyramid to the radius of the ball which is tangent to the base of the pyramid and the extensions of the lateral sides.

Problems for independent study

3.39. Two opposite vertices of the cube coincide with the centers of the bases of a cylinder and its other vertices lie on the lateral surface of the cylinder. Find the ratio of volumes of the cylinder and the cube.

3.40. Inside a prism of volume V a point O is taken. Find the sum of volumes of the pyramids with vertex O whose bases are lateral faces of the prism.

3.41. In what ratio the volume of the cube is divided by the plane passing through one of the cubes vertices and the centers of the two faces that do not contain this vertex?

3.42. Segment EF does not lie in plane of the parallelogram $ABCD$. Prove that the volume of tetrahedron $EFAD$ is equal to either sum or difference of volumes of tetrahedrons $EFAB$ and $EFAC$.

3.43. The lateral faces of an n-gonal pyramid are lateral faces of regular quadrangular pyramids. The vertices of the bases of quadrangular pyramids distinct from the vertices of an n-gonal pyramid pairwise coincide. Find the ratio of volumes of the pyramids.

3.44. The dihedral angle at edge AB of tetrahedron $ABCD$ is a right one; M is the midpoint of edge CD. Prove that the area of triangle AMB is a half area of the parallelogram whose diagonals are equal to and parallel to edges AB and CD.

3.45. Faces ABD, BCD and CAD of tetrahedron $ABCD$ serve as lower bases of the three prisms; the planes of their upper bases intersect at point P. Prove that the sum of volumes of these three prisms is equal to the volume of the prism whose base is face ABC and the lateral bases are equal and parallel to segment PD.

3.46. A regular tetrahedron of volume V is rotated through an angle of α ($0 < \alpha < \pi$) around a line that connects the midpoints of its skew edges. Find the volume of the common part of the initial tetrahedron and the rotated one.

3.47. A cube with edge a is rotated through the angle of α about the diagonal. Find the volume of the common part of the initial cube and the rotated one.

3.48. The base of a quadrilateral pyramid $SABCD$ is square $ABCD$ with side a. The angles between the opposite lateral faces are right ones; and the dihedral angle at edge SA is equal to α. Find the volume of the pyramid.

Solutions

3.1. Let h and h' be the lengths of perpendiculars dropped from points D and D' to plane ABC; let S and S' be the areas of triangles ABC and $AB'C'$. Clearly, $h : h' = AD : AD'$ and $S : S' = (AB \cdot AC) : (AB' \cdot AC')$. It remains to notice that

$$V_{ABCD} : V_{AB'C'D'} = hS : h'S'.$$

3.2. The height of triangle ABD dropped from vertex B is equal to $AB \sin\gamma$ and, therefore, the height of the tetrahedron dropped to plane ACD is equal to $AB \sin\gamma \sin D$. It is also clear that the area of triangle ACD is equal to $\frac{1}{2}AC \cdot AD \sin\beta$.

3.3. Let h_1 and h_2 be heights of the given faces dropped to their common side. Then

$$V = \frac{1}{3}(h_1 \sin\alpha)S_2 = \frac{ah_1h_2 \sin\alpha}{6}.$$

It remains to notice that $h_1 = \frac{2S_1}{a}, h_2 = \frac{2S_2}{a}$.

3.4. Consider the parallelepiped formed by planes passing through the edges of the tetrahedron parallel to the opposite edges. The planes of the faces of the initial tetrahedron cut off the parallelepiped four tetrahedrons; the volume of each of these tetrahedrons is $\frac{1}{6}$ of the volume of the parallelepiped. Therefore, the volume of the tetrahedron constitutes $\frac{1}{3}$ of the volume of the parallelepiped. The volume of the parallelepiped can be easily expressed in terms of the initial data: its face is a parallelogram with diagonals of length AB and CD and angle φ between them and the height dropped to this face is equal to d.

3.5. The angle α between line KO and height h of the pyramid is equal to the angle between the plane of the base and the plane perpendicular to KO. Hence, $h = KO\cos\alpha$ and $S = S'\cos\alpha$, where S' is the area of the base (cf. Problem 2.13). Therefore, $S \cdot KO = S'h$.

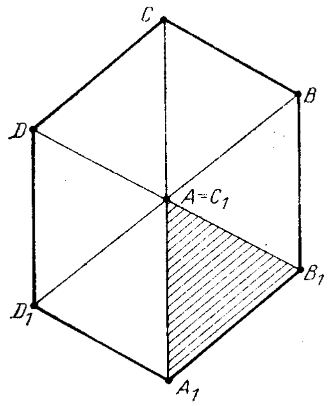

FIGURE 25 (3.6)

3.6. Consider the projection of the given parallelepiped to the plane perpendicular to line AC_1 (Fig. 25). In what follows in this solution we make use of notations from Fig. 25.

On this figure the lengths of segments AA_1, AB and AD are equal to distances from vertices A_1, B and D of the parallelepiped to the diagonal AC_1 and the sides of triangle AA_1B_1 are equal to these segments. Since the area of this triangle is equal to S, the area of triangle A_1DB is equal to $3S$. If M is the intersection point of plane A_1DB with diagonal AC_1, then $AM = \frac{1}{3}d$ (Problem 2.1) and, therefore, by Problem 3.5 the volume of tetrahedron AA_1DB is equal to $\frac{1}{3}dS$. It is also clear that the volume of this tetrahedron constitutes $\frac{1}{6}$ of the volume of the parallelepiped.

3.7. Let us connect the center of the sphere with the vertices of the polyhedron and, therefore, divide the polyhedron into pyramids. The heights of these pyramids are equal to the radius of the sphere and the faces of the polyhedron are their bases. Therefore, the sum of volumes of these pyramids is equal to $\frac{1}{3}SR$, where S is the sum of areas of their bases, i.e., the surface area of the polyhedron.

3.8. Both the cone and the sphere itself can be considered as a limit of polyhedrons circumscribed about the given sphere. It remains to notice that for each of these polyhedrons the formula $V = \frac{1}{3}SR$ holds, where V is the volume, S the surface area of the polyhedron and R the radius of the given sphere (Problem 3.7) holds.

3.9. The arguments literally the same as in the proof of Problem 3.8 show that the volume of this body is equal to $\frac{1}{3}SR$.

3.10. a) Consider an arbitrary section parallel to the bases. Let MP be the radius of the section of the cone, MC the radius of the section of the ball, MB the radius of the section of the cylinder. We have to verify that

$$\pi MP^2 = \pi MB^2 - \pi MC^2, \quad \text{i.e.,} \quad MB^2 = MP^2 + MC^2.$$

To prove this equality it suffices to notice that $MB = OC$, $MP = MO$ and triangle COM is a right one.

b) Volumes of the cylinder and the cone considered in heading a) are equal to πR^3 and $\frac{1}{3}\pi R^3$, respectively. The volume of the ball of radius R is twice the difference, of volumes of the cylinder and the cone, hence, it is equal to $\frac{4}{3}\pi R^3$.

3.11. The given cone is obtained by cutting off the cone with height x and the radius r of the base from the cone with height $x+h$ and the radius R of the base. Therefore,

$$V = \frac{\pi(R^2(x+h) - r^2x)}{3}.$$

Since $x : r = (x+h) : R$, then $x = \frac{rh}{R-r}$ and $x+h = \frac{Rh}{R-r}$; hence,

$$V = \frac{\pi(r^2 + rR + R^2)h}{3}.$$

3.12. First, suppose that the given planar figure is a convex n-gon. Then the considered body consists of a prism of volume $2dS$, n half cylinders with total volume πpd^2 and n bodies from which one can compose a ball of volume $\frac{4}{3}\pi d^3$. Let us describe the latter n bodies in detail. Consider a ball of radius d and cut it by semidisks (with centers at the center of the ball) obtained by shifts of the bases of semicylinders. This is the partition of the ball into n bodies.

Thus, if a figure is a convex polyhedron, then the volume of the body is equal to

$$2dS + \pi pd^2 + \frac{4}{3}\pi d^3.$$

This formula remains true for an arbitrary convex figure.

3.13. As in the preceding problem, let us divide the obtained body into the initial polyhedron, prisms corresponding to faces, the parts of cylinders corresponding to edges, and the parts of the ball of radius d corresponding to vertices. It is now easy to verify that the volume of the obtained body is equal to

$$V + Sd + \frac{1}{2}d^2\sum_i(\pi - \varphi_i)l_i + \frac{4}{3}\pi d^3$$

and the total surface of its area is equal to

$$S + d\sum_i(\pi - \varphi_i)l_i + 4\pi d^2.$$

3.14. First solution. Let O be the inner point of the polyhedron equidistant from the given planes. The area of the polyhedron confined between the given planes can be separated into triangles with vertices in the vertices of the polyhedron. Therefore, the polyhedron is divided into two pyramids with vertex O whose bases are the faces with areas S_1 and S_2 and several triangular pyramids with vertex O whose bases are the indicated triangles. The volumes of the first two pyramids are equal to $\frac{1}{6}hS_1$ and $\frac{1}{6}hS_2$. The volume of the i-th triangular pyramid is equal to $\frac{1}{3}2hs_i$, where s_i is the area of the section of this pyramid by the plane equidistant from the given ones; indeed the volume of the pyramid is 4 times the volume of the tetrahedron that the indicated plane cuts off it and the volume of the tetrahedron is equal to $\frac{1}{6}hs_i$. It is also clear that $s_1 + \dots + s_n = S$.

Second solution. Let $S(t)$ be the area of the section of the polygon by the plane whose distance from the first plane is equal to t. Let us prove that $S(t)$ is a quadratic function (for $0 \le t \le h$), i.e., that

$$S(t) = at^2 + bt + c.$$

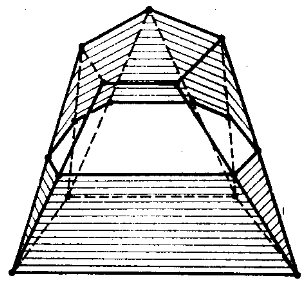

FIGURE 26 (SOL. 3.14)

To this end, consider the projection of the polyhedron to the first plane along a line chosen so that the projections of the upper and the lower faces do not intersect (Fig. 26). The areas of both shaded parts are quadratic functions in t; hence, $S(t)$ — the area of the unshaded part — is also a quadratic function.

For any quadratic function $S(t)$, where t runs from 0 to h, we can select a sufficiently simple polyhedron with exactly the same function $S(t)$:

if $a > 0$ we can take a truncated pyramid;

if $a < 0$ we can take the part of the tetrahedron confined between two planes parallel to two of its skew edges.

The volumes of polyhedrons with equal functions $S(t)$ are equal (by *Cavalieri's principle*). It is easy to verify that any of the new simple polyhedrons can be split into tetrahedrons whose vertices lie in given planes.

For them the required formula is easy to verify (if two vertices of a tetrahedron lie in one plane and the other two vertices lie in another plane we have to make use of the formula from Problem 3.4).

3.15. The volume of such a tetrahedron is equal to $\frac{1}{6}abd\sin\varphi$, where a and b are the lengths of the edges, d is the distance between skew lines and φ is the angle between them (Problem 3.4).

3.16. The projection to the plane perpendicular to given lines sends a, b and c into points A, B and C, respectively. Let s be the area of triangle ABC; KS the edge of the tetrahedron moving along line a. By Problem 3.5 the volume of the considered tetrahedron is equal to $\frac{1}{3}sKS$.

3.17. Let plane Π intersect the axis of the cylinder at point O. Let us draw through O plane Π' parallel to the basis of the cylinder. The planes Π and Π' divide the cylinder into 4 parts; of these, the two parts confined between the planes Π and Π' are of equal volume. Therefore, the volumes of the parts into which the cylinder is divided by plane Π are equal to the volumes of the parts into which it is divided by plane Π'. It is also clear that the ratio of the volumes of cylinders with equal bases is equal to the ratio of their heights.

3.18. Let M and K be the midpoints of edges AB and CD of tetrahedron $ABCD$. Let, for definiteness, the plane passing through M and K intersect edges AD and BC at points L and N (Fig. 27). Plane DMC divides the tetrahedron into two parts of equal volume, consequenlty, it suffices to verify that the volumes of tetrahedrons $DKLM$ and $CKNM$ are equal. The volume of tetrahedron $CKBM$ is equal to $\frac{1}{4}$ of the volume of tetrahedron $ABCD$ and the ratio of the volumes of tetrahedrons $CKBM$ and $CKNM$ is equal to $BC : CN$. Similarly, the ratio of a

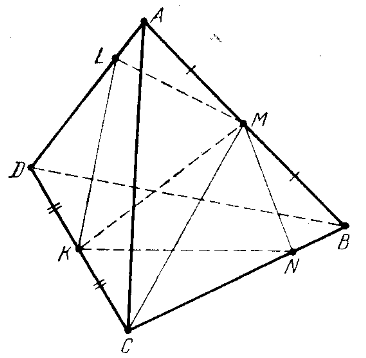

FIGURE 27 (SOL. 3.18)

quarter of the volume of tetrahedron $ABCD$ to the volume of tetrahedron $DKLM$ is equal to $AD : DL$. It remains to notice that $BC : CN = AD : DL$ (Problem 2.5).

3.19. By Problem 3.16 $V_{A'ABC} = V_{AA'B'C'}$. Writing down similar equalities for the volumes of tetrahedrons $A'ADC$ and $A'ABD$ and expressing $V_{A'BCD}$ and $V_{AB'C'D'}$ in terms of these volumes we get the statement desired.

3.20. Let A_2 be the intersection point of line AA_1 with plane $B_1C_1D_1$. Let us prove that $A_1A_2 = 3A_1A$. Then $V_{ABCD} : V_{A_2BCD} = 1 : 3$ and making use of the result of Problem 3.19 we finally get

$$V_{ABCD} : V_{A_1B_1C_1D_1} = V_{ABCD} : V_{A_2BCD} = 1 : 3.$$

Among the colinear vectors $\{BB_1\}$, $\{CC_1\}$ and $\{DD_1\}$ there are two directed similarly; for definiteness, assume that these are $\{BB_1\}$ and $\{CC_1\}$. Let M be the intersection point of lines BC_1 and CB_1. Lines BC_1 and CB_1 belong to planes ADB and ADC, respectively, hence, point M belongs to line AD.

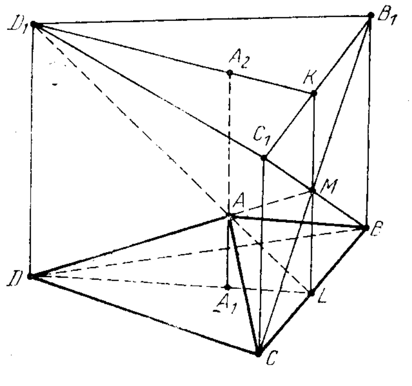

FIGURE 28 (SOL. 3.20)

Let us draw plane through parallel lines AA_1 and DD_1; it passes through point M and intersects segments BC and B_1C_1 at certain points L and K (Fig. 28). It is easy to verify that M is the midpoint of segment KL, point A belongs to lines DM and D_1L, point A_1 belongs to line DL, point A_2 belongs to line D_1K. Hence,

$$\{A_1A\} : \{AA_2\} = \{LM\} : \{LK\} = 1 : 2$$

and, therefore, $A_1A_2 = 3AA_1$.

3.21. Let P and Q be the midpoints of segments AC_1 and BC_1, respectively, i.e., PQ be the intersection line of the given planes. The ratio of volumes of tetrahedrons C_1PQC and C_1ABC is equal to

$$(C_1P : C_1A)(C_1Q : C_1B) = 1 : 4$$

(see Problem 3.1). It is also clear that the volume of tetrahedron C_1ABC constitutes $\frac{1}{3}$ of the volume of the prism. Making use of this fact, it is easy to verify that the desired ratio of volumes is equal to 1:3:3:5.

3.22. The common part of the indicated tetrahedrons is a convex polyhedron with vertices at the centers of the faces of the parallelepiped. The plane equidistant from two opposite faces of the parallelepiped cuts this polyhedron into two quadrangular pyramids the volume of each of which is equal to $\frac{1}{12}V$.

3.23. The section of the tetrahedron with the given plane is a parallelogram. Each of the two obtained parts of the tetrahedron can be divided into a pyramid, whose base is this parallelogram, and a tetrahedron. The volumes of these pyramids and tetrahedrons can be expressed through the lengths a and b of the skew edges, the distance d between them and angle φ (for tetrahedrons one has to make use of the formula from Problem 3.4). Thus, we find that the volumes of the obtained parts are equal to $\frac{10v}{81}$ and $\frac{7v}{162}$, where $v = abd \sin\varphi$, and the ratio of the volumes is equal to $\frac{20}{7}$.

3.24. On the extension of edge BB_1 beyond point B_1 mark segment B_1B_2 equal to edge AA_1. Let K be the midpoint of segment A_1B_1, i.e., the intersection point of segments A_1B_1 and AB_2. Since the volumes of tetrahedra A_1KC_1A and $B_1KC_1B_2$ are equal, the volumes of polyhedrons $ABCA_1B_1C_1$ and $ABCB_2C_1$ are also equal. Similar arguments show that the volume of polyhedron $ABCB_2C_1$ is equal to the volume of pyramid $ABCC_3$, where $CC_3 = AA_1 + BB_1 + CC_1$. It remains to make use of the formula from Problem 3.5.

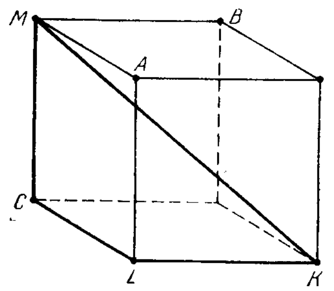

FIGURE 29 (SOL. 3.24)

3.25. Let us complete pyramid $MABC$ to a parallelepiped (see Fig. 29). Let MK be the diagonal of the parallelepiped. Since

$$\{MA\} + \{MB\} + \{MC\} + \{MD\} = \{0\}$$

(see Problem 14.3 a)), then $\{KM\} = \{MD\}$. Therefore, quadrilateral $MCLK$ is the one to be found. The volumes of tetrahedrons $MCKL$ and $MABC$ are equal, because each of them constitutes $\frac{1}{6}$ of the volume of the considered parallelepiped. It is also clear that the volume of tetrahedron $MABC$ is equal to $\frac{1}{4}V$.

Remark. It follows from the solution of Problem 7.15 that the collection of vectors of the sides of the required spatial quadrilateral is uniquely determined. Therefore, there exist 6 distinct such quadrilaterals and the volumes of all the tetrahedrons determined by them are equal (cf. Problem 8.26).

3.26. First, notice that after the rotation (in plane) of the segment of length $2d$ about a point that lies on the midperpendicular to this segment at distance x from the segment we get an annulus with the inner radius x and the outer radius $\sqrt{x^2+d^2}$; the area of this annulus is equal to πd^2, i.e., it does not depend on x. Hence, the section of the given body by the plane perpendicular to the axis of rotation is an annulus whose area does not depend on the position of line l. Therefore, it suffices to consider the case when the axis of rotation is the height of the triangle. In this case the volume of the body of rotation – the cone – is equal to $\frac{\pi a^3\sqrt{3}}{24}$.

3.27. Let $AC = x$, $BD = y$; let D_1 be the projection of D to the plane tangent to the ball at point A. In triangle CAD_1, angle $\angle A$ is equal to either α or $180^\circ - \alpha$ hence,

$$x^2 + y^2 \mp 2xy\cos\alpha = CD_1^2 = 4R^2\tan^2\varphi.$$

It is also clear that

$$x + y = CD = \frac{2R}{\cos\varphi}.$$

Therefore, either $xy = \frac{R^2}{\cos^2\frac{\alpha}{2}}$ or $xy = \frac{R^2}{\sin^2\frac{\alpha}{2}}$. Taking into account that $(x+y)^2 \geq 4xy$ we see that the first solution is possible for $\varphi \geq \frac{\alpha}{2}$ and the second one for $\varphi \geq \frac{1}{2}(\pi - \alpha)$. Since the volume V of tetrahedron $ABCD$ is equal to $\frac{1}{3}xyR\sin\alpha$, the final answer is as follows:

$$V = \begin{cases} \frac{2}{3}R^3\tan\frac{\alpha}{2} & \text{if } \alpha \leq 2\varphi < \pi - \alpha \\ \text{either } \frac{2}{3}R^3\tan\frac{\alpha}{2} \text{ or } \frac{2}{3}R^3\cot\frac{\alpha}{2} & \text{if } \pi - \alpha \leq 2\varphi < \pi. \end{cases}$$

3.28. On Figures 30 a)–d) the common parts of the pyramids in all the four cases are plotted.

a) The common part is a parallelepiped (Fig. 30 a)). This parallelepiped is obtained from the initial pyramid by cutting off the three pyramids similar to it with coefficient $\frac{2}{3}$; the three pyramids similar to the initial one with coefficient $\frac{1}{3}$ are common ones for the pairs of pyramids that are cut off. Hence, the volume of the pyramid is equal to

$$V(1 - 3(\frac{2}{3})^2 + 3(\frac{1}{3})^3) = \frac{2V}{9}.$$

b) The common part is an "octahedron" (Fig. 30 b)). The volume of this polyhedron is equal to $V(1 - 4(\frac{1}{2})^3) = \frac{1}{2}V$.

c) The common part is depicted on Fig. 30 c). To compute its volume, we have to subtract from the volume of the initial pyramid the volume of the pyramid similar to it with coefficient $\frac{1}{3}$ (on the figure this smaller pyramid is the one above)

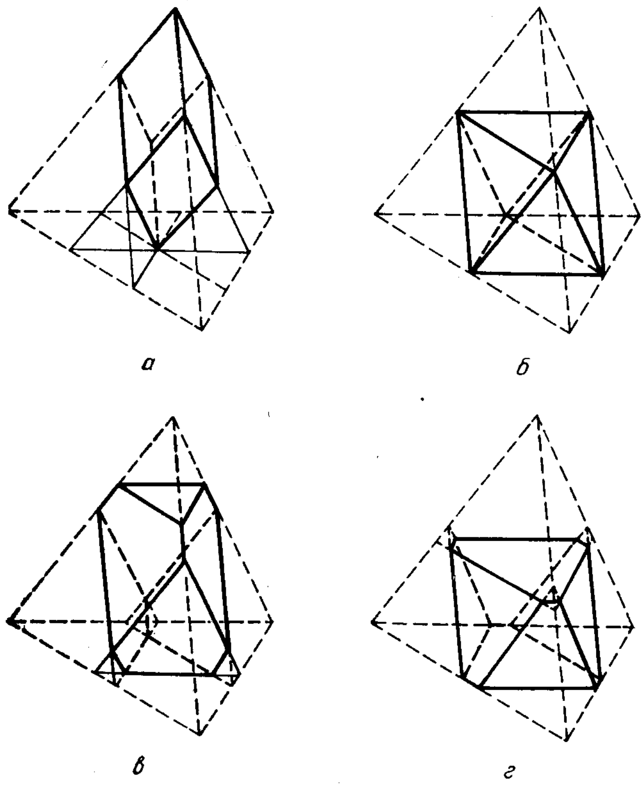

FIGURE 30 (SOL. 3.28)

then subtract the volume of three pyramids similar to the initial one with coefficient $\frac{5}{9}$ and add the volume of three pyramids similar to the initial one with coefficient $\frac{1}{9}$. Therefore, the volume of the common part is equal to

$$V(1-(\frac{1}{3})^3-3(\frac{5}{9})^3+3(\frac{1}{9})^3)=\frac{110V}{243}.$$

d) The common part is depicted on Fig. 30 d). Its volume is equal to

$$V(1-(\frac{3}{5})^3-3(\frac{7}{15})^3+3(\frac{1}{15})^3)=\frac{12V}{25}.$$

3.29. The existence of such a special quadrilateral $KLMN$ for any tetrahedron $ABCD$ follows from the statement of Problem 7.19; there are several such quadrilaterals but the volumes of all the tetrahedrons determined by them are equal (Problem 8.26).

Making use of the formula of Problem 3.2 it is easy to prove that

$$V^3=(\frac{abc}{6})^3p^2q,$$

where a, b and c are the lengths of the edges coming out of vertex A; p the product of the sines of the plane angles at vertex A; q the product of the sines of dihedral angles of the trihedral angle at vertex A. From an arbitrary point O from inside tetrahedron $ABCD$ drop perpendiculars to faces intersecting at A and depict on these perpendiculars segments OP, OQ and OR whose length measured in the chosen linear units is equal to the areas of the respective faces computed in the corresponding area units. It follows from the solution of Problem 8.26 that the volume W of tetrahedron $OPQR$ is equal to the volume of tetrahedron $KLMN$. The plane (resp. dihedral) angles of the trihedral angle $OPQR$ complement the dihedral (resp. planar) angles of the trihedral angle $ABCD$ to 180° (cf. Problem 5.1). Hence, $W^3 = (\frac{S_1S_2S_3}{6})^3q^2p$, where S_1, S_2, S_3 are the areas of the faces intersecting at vertex A. Since $S_1S_2S_3 = \frac{(abc)^2p}{8}$, it follows that

$$W^3 = (\frac{1}{6})^3(\frac{1}{8})^3(abc)^6p^4q^2 = (\frac{3}{4}V^2)^3, \quad \text{i.e.,} \quad W = \frac{3}{4}V^2.$$

3.30. Let M and N be the midpoints of edges B_1C_1 and BC, respectively. The considered pairs of planes are symmetric through plane AA_1MN. On ray MN take point K so that $MK = 2MN$. Since AA_1MN is a square, then $KA \perp AM$; hence, line AK is perpendicular to plane AB_1C_1, i.e., AK is the intersection line of the considered planes passing through point A.

We similarly construct the intersection line A_1L of planes passing through point A_1. Since B_1N is the projection of line AB_1 to plane BCC_1, the plane passing through point A perpendicularly to AB_1 intersects plane BCC_1 along the line perpendicular to line B_1N. After similar arguments for the other considered planes and taking into account that triangles BMC and B_1NC_1 are equilateral ones, we see that the obtained planes cut off the plane BCC_1B_1 a rhombus consisting of two equilateral triangles with side $KL = 3a$. The area of this rhombus is equal to $\frac{9\sqrt{3}}{2}a^2$. The figure to be constructed is a quadrilateral pyramid with this rhombus as its base and the intersection point S of lines AK and A_1L as its vertex. Since the distance from S to line KL is equal to $\frac{3}{2}a$, the volume of this pyramid is equal to $\frac{9\sqrt{3}}{4}a^3$.

3.31. Let K, L and M be the midpoints of segments AB, AC and AD, respectively. First, let us prove that K is the midpoint of segment DC_1. Point B lies in plane $A_1C_1D_1$; hence, point C_1 lies in plane A_1LB. Let us complement tetrahedron $ABCD$ to a triangular prism by adding vertices S and T, where $\{AS\} = \{DB\}$ and $\{AT\} = \{DC\}$. Plane A_1LB passes through the midpoints of sides CD and AT of parallelogram $CDAT$; hence, it contains line BS. Therefore, S is the intersection point of line DK with plane A_1LB, i.e., $S = C_1$.

We similarly prove that L and M are the midpoints of segments BD_1 and CB_1. Thus, tetrahedron $A_1B_1C_1D_1$ is bounded by planes A_1LB, A_1MC and A_1KD and plane $B_1C_1D_1$ passing through point A parallel to face BCD.

Let Q be the midpoint of BC, P the intersection point of BL and KQ (Fig. 31). Plane A_1KD cuts off tetrahedron $ABCD$ a tetrahedron $DKBQ$ whose volume is equal to $\frac{1}{4}V$. Planes A_1LB and A_1MC cut off tetrahedrons of the same volume.

For tetrahedrons cut off by planes A_1KD and A_1LB the tetrahedron A_1BPQ whose volume is equal to $\frac{1}{24}V$ is a common one. Therefore, the volume of the common part of tetrahedrons $ABCD$ and $A_1B_1C_1D_1$ is equal to

$$V(1 - \frac{3}{4} + \frac{3}{24}) = \frac{3V}{8}.$$

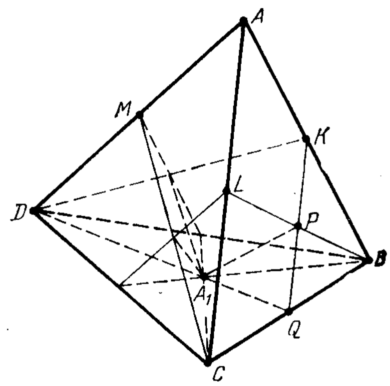

FIGURE 31 (SOL. 3.31)

3.32. The ratio of the segments of the edge is equal to the ratio of the heights dropped from its endpoints to the bisector plane and the latter ratio is equal to the ratio of volumes of tetrahedrons into which the bisector plane divides the given tetrahedron. Since the heights dropped from any point of the bisector plane to the faces of the dihedral angle are equal, the ratio of the volumes of these tetrahedrons is equal to the ratio of areas of the faces that confine the given dihedral angle.

3.33. Let $a = AB$, x be the area of the section to be constructed. Making use of the formula from Problem 3.3 for the volume of tetrahedron $ABCD$ and its parts we get

$$\frac{2}{3}\frac{px\sin(\frac{\alpha}{2})}{a} + \frac{2}{3}\frac{qx\sin(\frac{\alpha}{2})}{a} = \frac{2}{3}\frac{pq\sin\alpha}{a}.$$

Hence, $x = \frac{2pq}{p+q}\cos\frac{\alpha}{2}$.

3.34. Let us divide the tetrahedron into 4 triangular pyramids whose bases are the tetrahedron's faces and the vertex is at the given point. The indicated sum of ratios is the sum of ratios of the volumes of these pyramids to the volume of the tetrahedron. This sum is equal to 1 since the sum of volumes of the pyramids is equal to the volume of the tetrahedron.

3.35. Parallel segments AD and OA_1 form equal angles with plane BCD, consequently, the ratio of the lengths of the heights dropped to this plane from points O and A is equal to the ratio of lengths of these segments. Hence, $\frac{V_{OBCD}}{V_{ABCD}} = \frac{OA_1}{DA}$.

Writing similar equalities for segments OB_1 and OC_1 and adding them we get

$$\frac{OA_1}{DA} + \frac{OB_1}{DB} + \frac{OC_1}{DC} = \frac{V_{OBCD} + V_{OACD} + V_{OABD}}{V_{ABCD}} = 1.$$

3.36. Let S_a, S_b, S_c and S_d be the areas of faces BCD, ACD, ABD and ABC; V the volume of the tetrahedron; O the center of the sphere tangent to face BCD and the extensions of the other three faces. Then

$$3V = r_a(-S_a + S_b + S_c + S_d).$$

Hence,

$$\frac{1}{r_a} = \frac{-S_a + S_b + S_c + S_d}{3V}.$$

Writing similar equalities for the other radii of the escribed spheres and adding them, we get

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}+\frac{1}{r_d}=\frac{2(S_a+S_b+S_c+S_d)}{3V}=\frac{2}{r}.$$

3.37. It is possible to cut pyramid $MA_1B_1C_1D_1$ into two tetrahedrons by plane MA_1C_1 as well as by plane MB_1D_1, hence,

$$V_{MB_1C_1D_1}+V_{MA_1B_1D_1}=V_{MA_1B_1C_1}+V_{MA_1C_1D_1}. \tag{1}$$

Making use of formulas from Problem 3.1 we get

$$V_{MB_1C_1D_1}=\frac{MB_1}{MB}\cdot\frac{MC_1}{CM}\cdot\frac{MD_1}{MD}V_{MBCD}=$$
$$\frac{1}{3}h\left(\frac{MA_1}{MA}\cdot\frac{MB_1}{MB}\cdot\frac{MC_1}{MC}\cdot\frac{MD_1}{MD}\right)\frac{MA}{MA_1}S_{BCD},$$

where h is the height of pyramid $MABCD$. Substituting similar expressions for the volumes of all the other tetrahedrons into (1) we get the desired statement after simplification.

3.38. Let r and r' be the radii of the circumscribed and escribed balls, respectively, S the area of the lateral face, s the area of the base, V the volume of the pyramid. Then $V=\frac{(3S+s)r}{3}$. We similarly prove that

$$V=\frac{(3S-s)r'}{3}.$$

Moreover,

$$s=(\cos\alpha+\cos\beta+\cos\gamma)S$$

(cf. Problem 2.13). Hence,

$$\frac{r}{r'}=\frac{3S-s}{3S+s}=\frac{3-\cos\alpha-\cos\beta-\cos\gamma}{3+\cos\alpha+\cos\beta+\cos\gamma}.$$

CHAPTER 4. SPHERES

§1. The length of the common tangent

4.1. Two balls of radii R and r are tangent to each other. A plane is tangent to these balls at points A and B. Prove that $AB = 2\sqrt{Rr}$.

4.2. Three balls are tangent pairwise; a plane is tangent to these balls at points A, B and C. Find the radii of these balls if the sides of triangle ABC are equal to a, b and c.

4.3. Two balls of the same radius and two balls of another radius are placed so that each ball is tangent to the three other ones and a given plane. Find the ratio of the balls' radii.

4.4. The radii of two nonintersecting balls are equal to R and r; the distance between their centers is equal to a. Between what limits can the length of the common tangent to these balls vary?

4.5. Two tangent spheres are inscribed in a dihedral angle of value 2α. Let A be the tangent point of the first sphere with the first face and B the tangent point of the second sphere with the second face. What is the ratio into which segment AB is divided by the intersection points with these spheres?

§2. Tangents to the spheres

4.6. From an arbitrary point in space perpandiculars to planes of the faces of the given cube are dropped. The obtained segments are diagonals of six other cubes. Let us consider six spheres each of which is tangent to all the edges of the corresponding cube. Prove that all these spheres have a common tangent line.

4.7. A sphere with diameter CE is tangent to plane ABC at point C; line AD is tangent to the sphere. Prove that if point B lies on line DE, then $AC = AB$.

4.8. Given cube $ABCDA_1B_1C_1D_1$. A plane passing through vertex A and tangent to the sphere inscribed in the cube intersects edges A_1B_1 and A_1D_1 at points K and N, respectively. Find the value of the angle between planes AC_1K and AC_1N.

4.9. Two equal triangles KLM and KLN have a common side KL, moreover, $\angle KLM = \angle LKN = 60^\circ$, $KL = 1$ and $LM = KN = 6$. Planes KLM and KLN are perpendicular. Find the radius of the ball tangent to segments LM and KN at their midpoints.

4.10. All the possible tangents to the given sphere are drawn from points A and B. Prove that all the intersection points of these tangents distinct from A and B lie in two planes.

4.11. The centers of three spheres whose radii are equal to 3, 4 and 6 lie in the vertices of an equilateral triangle with side 11. How many planes simultaneously tangent to all these spheres are there?

§3. Two intersecting circles lie on one sphere

4.12. a) Two circles not in one plane intersect at two distinct points, A and B. Prove that there exists a unique sphere that contains these circles.

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b) Two circles not in one plane are tangent to line l at point P. Prove that there exists a unique sphere containing these circles.

4.13. Given a truncated triangular pyramid, prove that if two of its lateral faces are inscribed quadrilaterals, then the third lateral face is also an inscribed quadrilateral.

4.14. All the faces of a convex polyhedron are inscribed polygons and all the angles are trihedral ones. Prove that around this polyhedron a sphere can be circumscribed.

4.15. Three spheres have a common chord. Through a point of this chord three chords belonging to distinct spheres are drawn. Prove that the endpoints of these three chords lie either on one sphere or in one plane.

4.16. Several circles are placed in space so that any two of them have a pair of common points. Prove that either all these circles have two common points or all of them belong to one sphere (or one plane).

4.17. Three circles in space are pairwise tangent to each other (i.e., they have common points and common tangents at these points) and all the three tangent points are distinct. Prove that either these circles belong to one sphere or to one plane.

§4. Miscellaneous problems

4.18. Three points A, B and C on a sphere of radius R are pairwise connected by (smaller) arcs of great circles. Through the midpoints of arcs $\smile AB$ and $\smile AC$ one more great circle is drawn; it intersects the continuation of arc $\smile BC$ at point K. Find the length of arc $\smile CK$ if the length of arc $\smile BC$ is equal to l ($l < \pi R$).

4.19. Chord AB of a unit sphere is of length 1 and constitutes an angle of 60° with diameter CD of this sphere. It is known that $AC = \sqrt{2}$ and $AC < BC$. Find the length of segment BD.

4.20. Given a sphere, a circle on it and a point P not on the sphere. Prove that the second intersection points of the sphere with the lines that connect point P with the points on the circle lie on one circle.

4.21. On a sphere of radius 2, we consider three pairwise tangent unit circles. Find the radius of the smallest circle lying on the given sphere and tangent to all the three given circles.

4.22. Introduce a coordinate system with the origin O at the center of the Earth, axes Ox and Oy passing through the points of equator with longitude 0° and 90°, respectively, and the Oz-axis passing through the North Pole. What are the coordinates on the surface of the Earth with latitude φ and longitude ψ? (We assume that the Earth is a ball of radius R; the latitude is negative in the southern hemisphere.)

4.23. Consider all the points on the surface of earth whose geographic latitude is equal to their longitude. Find the locus of the projections of these points to the plane of the equator.

§5. The area of a spherical band and the volume of a spherical segment

4.24. Two parallel planes the distance between which is equal to h cross a sphere of radius R. Prove that the surface area of the part of the sphere confined between them is equal to $2\pi Rh$.

4.25. Let A be the vertex of a spherical segment, B the point on the circle of its base. Prove that the surface area of this segment is equal to the area of the disk of radius AB.

4.26. Let h be the height of the spherical segment (Fig. 32), R the radius of the ball. Prove that the volume of the spherical segment is equal to $\frac{2\pi R^2 h}{3}$.

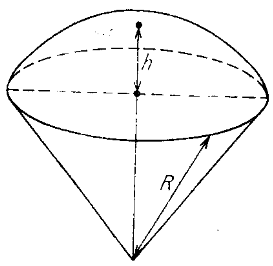

FIGURE 32 (4.26)

4.27. Let h be the height of the sperical segment and R the radius of the sphere, see Fig. 33. Prove that the volume of the sperical segment is equal to $\frac{1}{3}\pi h^2(2R-h)$.

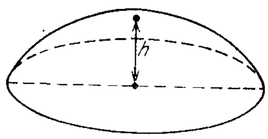

FIGURE 33 (4.27)

4.28. Prove that the volume of the body obtained after rotation of a circular segment about a diameter that does not intersect the segment is equal to $\frac{1}{6}\pi a^2 h$, where a is the length of the chord of this segment and h is the length of the projection of this chord to the diameter.

4.29. A golden ring is of the form of the body bounded by the surface of a ball and a cylinder (Fig. 34). How much gold should be added in order to increase k times the diameter d and preserving the height h?

4.30. The center of sphere S_1 belongs to sphere S_2 and it is known that the spheres intersect. Prove that the area of the part of the surface of S_2 situated inside S_1 is equal to $\frac{1}{4}$ of the surface area of S_1.

4.31. The center of sphere α belongs to sphere β. The area of the part of the surface of sphere β that lies inside α is equal to $\frac{1}{5}$ of the surface area of α. Find the ratio of the radii of these spheres.

4.32. A 20-hedron is circumscribed about a sphere of radius 10. Prove that on the surface of the 20-hedron there are two points the distance between which is greater than 21.

4.33. The length of a cube's edge is equal to a. Find the areas of the parts into which the planes of the cube's faces split the sphere circumscribed about the cube.

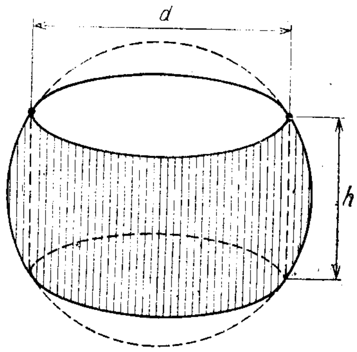

FIGURE 34 (4.29)

4.34. A ball of radius R is tangent to the edges of a regular tetrahedral angle (see §9.1) all the plane angles of which are equal to $60°$. The surface of the ball situated inside the angle consists of two curvilinear quadrilaterals. Find their areas.

4.35. Given a regular tetrahedron with edge 1, three of its edges coming out of one vertex and a sphere tangent to these edges at their endpoints. Find the area of the part of the sphere's surface confined inside the tetrahedron.

4.36. On a sphere of radius 2, lie three pairwise tangent circles of radius $\sqrt{2}$. The part of the sphere's surface outside the circles is the union of two curvilinear triangles. Find the areas of these triangles.

§6. The radical plane

Let line l passing through point O intersect a sphere S at points A and B. It is easy to verify that the product of the lengths of segments OA and OB only depends on O and S but does not depend on the choice of line l (for points that lie outside the sphere the product is equal to the squared length of the tangent's segment drawn from point O to the tangent point). This quantity taken with "plus" sign for points outside S and with "minus" sign for points inside S is called *the degree* of point O relative to sphere S. It is easy to verify that the degree of point O is equal to $d^2 - R^2$, where d is the distance from O to the center of the sphere and R is the radius of the sphere.

4.37. Given two nonconcentric spheres, prove that the locus of the points whose degrees relative to these spheres are equal is a plane.

This plane is called the *radical plane* of these two spheres.

4.38. Common tangents AB and CD are drawn to two spheres. Prove that the lengths of projections of segments AC and BD to the line passing through the centers of the spheres are equal.

4.39. Find the locus of the midpoints of common tangents to the two given nonintersecting spheres.

4.40. Inside a convex polyhedron, several nonintersecting balls of distinct radii are placed. Prove that this polyhedron can be cut into smaller convex polyhedra each of which contains exactly one of the given balls.

§7. The spherical geometry and solid angles

4.41. On a sphere, two intersecting circles S_1 and S_2 are given. Consider a cone (or a cylinder) tangent to the given sphere along circle S_1. Prove that circles S_1 and S_2 are perpendicular to each other if and only if the plane of S_2 passes through the vertex of this cone (or is parallel to the axis of the cylinder).

4.42. Find the area of a curvilinear triangle formed by the intersection of the sphere of radius R with the trihedral angle whose dihedral angles are equal to α, β and γ and the vertex coincides with the center of the sphere.

4.43. Let A_1 and B_1 be the midpoints of sides BC and AC of a spherical triangle ABC. Prove that the area of spherical triangle A_1B_1C is smaller than a half area of spherical triangle ABC.

4.44. A convex n-hedral angle cuts a spherical n-gon on the sphere of radius R with center at the vertex of the angle. Prove that the area of the spherical n-gon is equal to

$$R^2(\sigma - (n-2)\pi),$$

where σ is the sum of dihedral angles.

4.45. Two points, A and B, are fixed on a sphere. Find the locus of the third vertices C of spherical triangles ABC for which $\angle A + \angle B - \angle C$ is constant.

4.46. Two points A and B are fixed on a sphere. Find the locus of the third vertices C of spherical triangles ABC of given area.

4.47. Three arcs of great circles 300° each lie on a sphere. Prove that at least two of them have a common point.

4.48. Given several arcs of great circles on a sphere such that the sum of their angular values is smaller than π. Prove that there exists a plane passing through the center of the sphere and not intersecting either of these arcs.

Consider the unit sphere with the center in the vertex of a polyhedral angle (or on an edge of the dihedral angle). The area of the part of the sphere's surface confined inside this angle is called the value of the *solid angle* of this polyhedral (dihedral) angle.

4.49. a) Prove that the solid angle of the dihedral angle is equal to 2α, where α is the value of the dihedral angle in radians.

b) Prove that the solid angle of a polyhedral angle is equal to $\sigma - (n-2)\pi$, where σ is the sum of its dihedral angles.

4.50. Calculate the value of the solid angle of a cone with angle 2α at the vertex.

4.51. Prove that the difference between the sum of the solid angles of the dihedral angles of a tetrahedron and the sum of the solid angles of its trihedral angles is equal to 4π.

4.52. Prove that the difference between the sum of the solid angles of the dihedral angles at the edges of a polyhedron and the sum of the solid angles of the polyhedral angles at its vertices is equal to $2\pi(F-2)$, where F is the number of faces of the polyhedron.

Problems for independent study

4.53. Through point D, three lines intersecting a sphere at points A and A_1, B and B_1, C and C_1, respectively, are drawn. Prove that triangle $A_1B_1C_1$ is similar to the triangle with sides whose lengths measured in length units are equal to $AB \cdot CD$, $BC \cdot AD$ and $AC \cdot BD$ measured in the corresponding area units.

4.54. Consider the section of tetrahedron $ABCD$ with the plane perpendicular to the radius of the circumscribed sphere and with an endpoint at vertex D. Prove that 6 points — vertices A, B, C and the intersection points of the plane with edges DA, DB, DC — lie on one sphere.

4.55. Given cube $ABCDA_1B_1C_1D_1$ and the plane drawn through vertex A and tangent to the ball inscribed in the cube. Let M and N be the intersection points of this plane with lines A_1B and A_1D, respectively. Prove that line MN is tangent to the ball inscribed in the cube.

4.56. Consider a pyramid. A ball of radius R is tangent to all the pyramid's lateral faces of and at the midpoints of the sides of its bases. The segment which connects a vertex of the pyramid with the center of the ball is divided in halves by its intersection point with the base of the pyramid. Find the volume of the pyramid.

4.57. On a sphere, circles S_0, S_1, … , S_n are placed so that S_1 is tangent to S_n and S_2, S_2 is tangent to S_1 and S_3, … , S_n is tangent to S_{n-1} and S_1 and S_0 is tangent to all the circles. Moreover, the radii of all these circles are equal. For which n this is possible?

4.58. Let K be the midpoint of segment AA_1 of cube $ABCDA_1B_1C_1D_1$, let point L lie on edge BC so that segment KL is tangent to the ball inscribed in the cube. What is the ratio in which the tangent point divides segment KL?

4.59. The planes of a cone's base and its lateral surface are tangent from the inside to n pairwise tangent balls of radius R; n balls of radius $2R$ are similarly tangent to the lateral surface from the outside. Find the volume of the cone.

4.60. A plane intersects edges AB, BC, CD and DA of tetrahedron $ABCD$ at points K, L, M and N, respectively; P is an arbitrary point in space. Lines PK, PL, PM and PN intersect the circles circumscribed about triangles PAB, PBC, PCD and PDA for the second time at points K_1, L_1, M_1 and N_1, respectively. Prove that points P, K_1, L_1, M_1 and N_1 lie on one sphere.

Solutions

4.1. First, let us prove that the length of the common tangent to the two tangent circles of radii R and r is equal to $2\sqrt{Rr}$. To this end, let us consider a right triangle the endpoints of whose hypothenuse are the centers of circles and one of the legs is parallel to the common tangents. Applying to this triangle the Pythagoras' theorem we get

$$x^2 + (R-r)^2 = (R+r)^2,$$

where x is the length of the common tangent. Therefore, $x = 2\sqrt{Rr}$.

Now, by considering the section that passes through the centers of the given balls and points A and B it is easy to verify that this formula holds in our case as well.

4.2. Let x, y and z be the radii of the balls. By Problem 4.1, $a = 2\sqrt{xy}$, $b = 2\sqrt{yz}$ and $c = 2\sqrt{xz}$. Therefore, $\frac{ac}{b} = 2x$, i.e., $x = \frac{ac}{2b}$. Similarly, $y = \frac{ab}{2c}$ and $z = \frac{bc}{2a}$.

4.3. Let A and C be the tangent points of the balls of radius R with the plane; B and D be the tangent points of the balls of radius r with the plane. By Problem 4.1 $AB = BC = CD = AD = 2\sqrt{Rr}$; hence, $ABCD$ is a rhombus; its diagonals are equal to $2R$ and $2r$. Therefore, $R^2 + r^2 = 4Rr$, i.e., $R = (2 \pm \sqrt{3})r$. Consequently, the ratio of the large radius to the smaller one is equal to $2 + \sqrt{3}$.

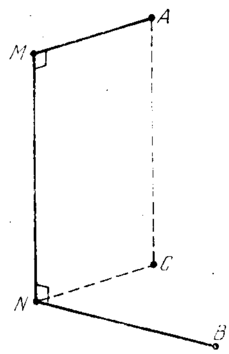

FIGURE 35 (SOL. 4.3)

4.4. Let MN be the common tangent, A and B the centers of the balls. The radii AM and BN are perpendicular to MN. Let C be the projection of point A to the plane passing through point N and perpendicular to MN (Fig. 35). Since $NB = r$ and $NC = R$, it follows that BC can vary from $|R-r|$ to $R+r$. Therefore, the value of

$$MN^2 = AC^2 = AB^2 - BC^2$$

can vary from $a^2 - (R^2 + r)^2$ to $a^2 - (R - r)^2$.

For the intersecting circles the upper limit of the length of MN is the same whereas the lower one is equal to 0.

4.5. Let a and b be the radii of spheres, A_1 and B_1 be the other tangent points with the faces of the angle. It is easy to compute the lengths of the sides of trapezoid AA_1BB_1; they are $AB_1 = A_1B = 2\sqrt{ab}$ (Problem 4.1), $AA_1 = 2a\cos\alpha$ and $BB_1 = 2b\cos\alpha$. The squared height of this trapezoid is equal to

$$4ab - (b-a)^2\cos^2\alpha$$

and the square of the diagonal is equal to

$$4ab - (b-a)^2\cos^2\alpha + (a+b)^2\cos^2\alpha = 4ab(1+\cos^2\alpha).$$

If the sphere that passes through points A and A_1 intersects segment AB at point K, then

$$BK = \frac{BA_1^2}{BA} = \frac{2\sqrt{ab}}{\sqrt{1+\cos^2\alpha}} = \frac{AB}{1+\cos^2\alpha}; \quad AK = \frac{AB\cos^2\alpha}{1+\cos^2\alpha}.$$

The lengths of the segments the intersection point with the second sphere divides AB into are similarly found. As a result, we see that segment AB is divided in the ratio $\cos^2\alpha : \sin^2\alpha : \cos^2\alpha$.

4.6. First, let us consider the given cube $ABCDA_1B_1C_1D_1$. The cone with axis AC_1 and generator AB is tangent to the sphere which is tangent to all the edges of the given cube. Therefore, the cone with axis AB and generator AC_1 is tangent to the sphere which is tangent to all the edges of the cube with diagonal AB. These arguments show that any of the four lines that pass through the given point parallel to any of the diagonals of the given cube is tangent to all the obtained spheres.

4.7. Since AC and AD are tangent to the given sphere, they are equal. Therefore, point A belongs to the plane passing through the midpoint of segment CD and perpendicular to it. Since $\angle CDB = 90^\circ$, this plane intersects plane ABC along the line passing through the midpoint of segment BC and perpendicular to it.

4.8. First, let us prove the following auxiliary statement. *Let two planes that intersect along line AX be tangent to the sphere with center O at points F and G. Then AOX is the bisector plane of the dihedral angle formed by planes AOF and AOG.* Indeed, points F and G are symmetric through plane AOX.

Let plane AKN be tangent at point P to the sphere inscribed in the cube and let line AP intersect NK at point M. Let us apply the statement proved above to the tangent planes passing through line NA. We see that AC_1N is the bisector plane of the dihedral angle formed by planes AC_1D_1 and AC_1M. Similarly, AC_1K is the bisector plane of the dihedral angle formed by planes AC_1M and AC_1B_1. Therefore, the angle between planes AC_1N and AC_1K is equal to a half the dihedral angle formed by the half planes AC_1D_1 and AC_1B_1. By considering the projection to the plane perpendicular to AC_1 we see that the dihedral angle formed by half planes AC_1D_1 and AC_1B_1 is equal to 120°.

4.9. Let O_1 and O_2 be the projections of the center O of the given ball to planes KLM and KLN, respectively; let P and S be the midpoints of segments LM and KN, respectively. Since $OP = OS$ and $PK = SL$, it follows that $OK = OL$. Therefore, the projections of points O_1 and O_2 to line KL coincide with the midpoint Q of segment KL. Since planes KLM and KLN are perpendicular to each other, $OO_1 = O_2Q = QO_1$; hence, the squared radius of the sphere to be found is equal to $PO_1^2 + OO_1^2 = PO_1^2 = QO_1^2$.

Applying the law of cosines to triangle KLM we get $KM^2 = 31$. By the law of sines $31 = (2R\sin 60^\circ)^2 = 3R^2$. Hence,

$$PO_1^2 + QO_1^2 = (R^2 - PL^2) + (R^2 - QL^2) = \frac{62}{3} - 9 - \frac{1}{4} = \frac{137}{12}.$$

4.10. Let O be the center of the given sphere, r its radius; a and b the lengths of tangents drawn from points A and B; let M be the intersection point of the tangents drawn from A and B; $let x$ be the length of the tangent drawn from M. Then $AM^2 = (a \pm x)^2$, $BM^2 = (b \pm x)^2$ and $OM^2 = r^2 + x^2$. Let us select numbers α, β and γ so that the expression

$$\alpha AM^2 + \beta BM^2 + \gamma OM^2$$

does not depend on x, i.e., so that $\alpha + \beta + \gamma = 0$ and $\pm 2\alpha a \pm 2\beta b = 0$. We see that point M satisfies either the relation

$$bAM^2 + aBM^2 - (a + b)OM^2 = d_1$$

or the relation

$$bAM^2 - aBM^2 + (a - b)OM^2 = d_2.$$

Each of these relations determines a plane, cf. Problem 1.29.

4.11. Let us consider a plane tangent to all the three given spheres and let us draw the plane through the center of the sphere of radius 3 parallel to the first plane. The obtained plane is tangent to spheres of radii 4 ± 3 and 6 ± 3 concentric to the spheres of radii 4 and 6.

If the signs of 3 are the same, the tangency is the outer one, and if they are distinct, the tangency is an inner one. It is also clear that for every plane tangent to all the spheres the plane symmetric to it through the plane passing through the centers of the spheres is also tangent to all the spheres.

In order to find out whether the plane passing through the given point and tangent to the two given spheres exists, we can make use of the result of Problem 12.11. In all the cases, except for the inner tangency with spheres of radius 1 and 9, the tangent planes exist (see Fig. 36).

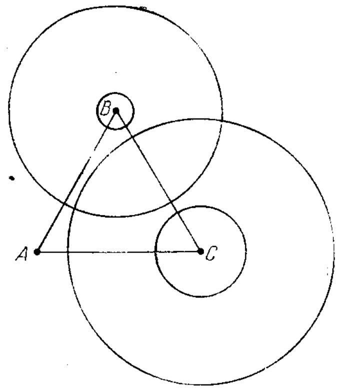

FIGURE 36 (SOL. 4.11)

Let us prove that there is no plane passing through point A and inner tangent to the spheres of radii 1 and 9 with centers B and C, respectively. Let α be the angle between line AB and the tangent from A to the sphere with center B; let β be the angle between line AC and the tangent from A to the sphere with center C. It suffices to verify that $\alpha + \beta > 60^\circ$, i.e., $\cos(\alpha+\beta) < \frac{1}{2}$.

Since $\sin\alpha = \frac{1}{11}$ and $\sin\beta = \frac{9}{11}$, it follows that $\cos\alpha = \frac{\sqrt{120}}{11}$ and $\cos\beta = \frac{\sqrt{40}}{11}$. Therefore, $\cos(\alpha+\beta) = \frac{40\sqrt{3}-9}{121}$. Thus, the inequality $\cos(\alpha+\beta) < \frac{1}{2}$ is equivalent to the inequality $80\sqrt{3} < 139$ and the latter inequality is verified by squaring.

As a result, we see that there are 3 pairs of tangent planes altogether.

4.12. Let O_1 and O_2 be the centers of the given circles; in heading a) M is the midpoint of segment AB and in heading b) $M = P$.

Consider plane MO_1O_2. The intersection point of perpendiculars erected in this plane from points O_1 and O_2 to lines MO_1 and MO_2 is the center of the sphere to be found.

4.13. The circumscribed circles of two of the lateral faces have two common points, the common vertices of these faces. Therefore, there exists the sphere that contains both of these circles. The circumscribed circle of the third face is the section of this sphere with the plane of the face.

4.14. Let us consider the vertex of the polyhedron and three more vertices — the endpoints of the edges that go out of it. It is possible to draw a sphere through these four points. Such spheres can be constructed for any vertex of the polyhedron and therefore, it suffices to prove that these spheres coincide for neighbouring vertices.

Let P and Q be some neighbouring vertices. Let us consider the circles circumscribed about two faces with common edge PQ. Point P and the endpoints of the three edges that go out of it belong to at least one of these circles.

The same is true for point Q. It remains to notice that through two circles not in one plane and with two common points and one can draw a sphere.

4.15. The product of the lengths of segments into which the intersection point divides each of the chords is equal to the product of the lengths of segments into which the common chord is divided by their intersection point, hence, these products are equal.

If segments AB and CD intersect at point O and $AO \cdot OB = CO \cdot OD$, then points A, B, C and D lie on one circle. Therefore, the endpoints of the first and second chords, as well as the endpoints of the second and third chords, lie on one circle. The second chord belongs to both of these circles; hence, these circles lie on one sphere.

4.16. If all the circles pass through some two points then all is proved. Therefore, we may assume that there are three circles such that the third circle does not pass through at least one of the intersection points of the first two circles. Let us prove then that these three circles lie on one sphere (or plane).

By Problem 4.12 a) the first two circles lie on one sphere (or plane). The third circle intersects the first circle at two points. These two points cannot coincide with the two intersection points of the third circle with the second one, because otherwise all the three circles would pass through two points. Hence, the third circle has at least three common points with the sphere determined by the first two circles. Therefore, the third circle belongs to this sphere.

Now, let us take some fourth circle. Its intersection points with the first circle can, certainly, coincide with the intersection points with the second circle, but then they cannot coincide with its intersection points with the third circle. Hence, the fourth circle has at least three common points with the sphere determined by the first two circles and, therefore, belongs to the sphere.

4.17. Let a sphere (or plane) α contain the first and the second circle, a sphere (or plane) β the second and the third circle. Suppose that α and β do not coincide. Then the second circle is the intersection curve. Moreover, the common point of the first and the third circles also belongs to the intersection curve of α and β, i.e., to the second circle, hence, all the three circles have a common point. Contradiction.

4.18. The plane that passes through the centers of the sphere and the midpoints of arcs $\smile AB$ and $\smile AC$ passes also through the midpoints of chords AB and AC and, therefore, is parallel to chord BC. Hence, the great circle passing through B and C and the great circle passing through the midpoints of arcs $\smile AB$ and $\smile AC$ intersect at points K and K_1 such that KK_1 is parallel to BC. Hence, the length of arc $\smile CK$ is equal to $\frac{1}{2}(\pi R \pm l)$.

4.19. Let O be the center of the sphere. Take point E so that $\{CE\} = \{AB\}$. Since $\angle OCE = 60^\circ$ and $CE = 1 = OC$, it follows that $OE = 1$. Point O is equidistant from all the vertices of parallelogram $ABEC$, hence, $ABEC$ is a rectangle and the projection O_1 of point O to the plane of this rectangle coincides with the rectangle's center, i.e., with the midpoint of segment BC. Segment OO_1 is a midline of triangle CBD, therefore,

$$BD = 2OO_1 = 2\sqrt{OC^2 - \frac{BC^2}{4}} = 2\sqrt{1 - \frac{AB^2 + AC^2}{4}} = 1.$$

4.20. Let A and B be two points of the given circle, A_1 and B_1 be the other intersection points of lines PA and PB with the sphere; l the tangent to the circle circumscribed about triangle PAB at point P. Then

$$\angle(l, AP) = \angle(BP, AB) = \angle(A_1B_1, AP),$$

i.e., $A_1B_1 \parallel l$. Let plane Π pass through point A_1 parallel to the plane tangent at P to the sphere that passes through the given point and P. All the desired points lie in plane Π.

4.21. Let O be the center of the sphere; O_1, O_2 and O_3 the centers of the given circles; O_4 the center of the circle to be found. By considering the section of the sphere with plane OO_1O_2, it is easy to prove that OO_1O_2 is a equilateral triangle with side $\sqrt{3}$. Line OO_4 passes through the center of triangle $O_1O_2O_3$ perpendicularly to the triangle's plane and, therefore, the distances from the vertices of this triangle to line OO_4 are equal to 1. Let K be the tangent point of the circles with centers O_1 and O_4; let L be the base of the perpendicular dropped from O_1 to OO_4; let N be the base of the perpendicular dropped from K to O_1L. Since $\triangle O_1KN \sim \triangle OO_1L$, it follows that $O_1N = \frac{OL \cdot O_1K}{OO_1} = \sqrt{\frac{2}{3}}$ and, therefore, the radius O_4K to be found is equal to $LN = 1 - \sqrt{\frac{2}{3}}$.

4.22. Let $P = (x, y, z)$ be the given point on the surface of the Earth, P' its projection to the equatorial plane. Then $z = R\sin\varphi$ and $OP' = R\cos\varphi$. Hence,

$$x = OP'\cos\psi = R\cos\varphi\cos\psi; \quad y = R\cos\varphi\sin\psi.$$

Thus, $P = (R\cos\varphi\cos\psi, R\cos\varphi\sin\psi, R\sin\varphi)$.

4.23. Introduce the same coordinate system as in Problem 4.22. If the latitude and the longitude of point P are equal to φ, then $P = (R\cos^2\varphi, R\cos\varphi\sin\varphi, R\sin\varphi)$. The coordinates of the projection of this point to the equatorial plane are $x = R\cos^2\varphi$ and $y = R\cos\varphi\sin\varphi$. It is easy to verify that

$$(x - \frac{R}{2})^2 + y^2 = \frac{R^2}{4},$$

i.e., the set to be found is the circle of radius $\frac{1}{2}R$ centered at $(\frac{1}{2}R, 0)$.

4.24. First, let us consider the truncated cone whose lateral surface is tangent to the ball of radius R and center O and let the tangent points divide the generators of the cone in halves. Let us prove that the area of the lateral surface of the cone is equal to $2\pi Rh$, where h is the height of the cone. Let AB be the generator of the truncated cone; C the midpoint of segment AB; let L be the base of the perpendicular dropped from C to the axis of the cone. The surface area of the truncated cone is equal to $2\pi CL \cdot AB$ (this formula can be obtained by the passage to the limit after we make use of the fact that the area of the trapezoid is equal to the product of its midline by the height) and, since the angle between line AB and the axis of the cone is equal to the angle between CO and CL, we have $AB : CO = h : CL$, i.e., $CL \cdot AB = CO \cdot h = Rh$.

Now the statement of the problem can be obtained by passage to the limit: let us replace the considered part of the spherical surface by a figure that consists from lateral surfaces of several truncated cones; when the heights of these cones tend to zero the surface area of this figure tends to the area of the considered part of the sphere.

4.25. Let M be the center of the base of the spherical segment, h the height of the segment, O the center of the ball, R the radius of the ball. Then $AM = h, MO = R - h$ and $BM \perp AO$. Hence,

$$AB^2 - AM^2 = BM^2 = BO^2 - OM^2,$$

i.e.,

$$AB^2 = h^2 + R^2 - (R-h)^2 = 2Rh.$$

It remains to make use of the result of Problem 4.24.

4.26. The volume of the spherical sector is equal to $\frac{2}{3}S$, where S is the area of the spherical part of the sector's surface. By Problem 4.24 $S = 2\pi Rh$.

4.27. A spherical segment together with the corresponding cone whose vertex is the center of the ball constitute a spherical sector. The volume of the spherical sector is equal to $\frac{2\pi R^2 h}{3}$ (Problem 4.26). The height of the cone is equal to $R - h$ and the squared radius of the cone's base is equal to

$$R^2 - (R-h)^2 = 2Rh - h^2;$$

consequently, the cone's volume is equal to $\frac{1}{3}\pi(R-h)(2Rh - h^2)$. By subtracting from the volume of the spherical sector the volume of the cone we get the statement desired.

4.28. Let AB be the chord of given segment, O the center of the disk, x the distance from O to AB, R the radius of the disk. The volume of the body obtained after rotation of the sector AOB about the diameter is equal to $\frac{1}{3}RS$, where S is the area of the surface obtained after rotation of arc $\smile AB$. By Problem 4.24 $S = 2\pi Rh$. From the solution of the same problem it follows that the volume of the body obtained after rotation of triangle AOB is equal to $\frac{2}{3}\pi x^2 h$ (to prove this, one has to observe that the part of the surface of this body obtained after rotation of segment AB is tangent to the sphere of radius x).

Thus, the desired volume is equal to

$$\frac{2\pi R^2 h}{3} - \frac{2\pi x^2 h}{3} = \frac{2\pi(x^2 + a^2/4)h}{3} - \frac{2\pi x^2 h}{3} = \frac{\pi a^2 h}{6}.$$

4.29. By Problem 4.28 the volume of the ring is equal to $\frac{1}{6}\pi h^3$, i.e., it does not depend on d.

4.30. Let O_1 and O_2 be the centers of spheres S_1 and S_2, let R_1 and R_2 be their radii. Further, let A be the intersection point of the spheres, AH the height of triangle O_1AO_2. Inside S_1 lies a segment of the sphere S_2 with height O_1H. Since $O_1O_2 = AO_2 = R_2$ and $O_1A = R_1$, it follows that $2O_1H : R_1 = R_1 : R_2$, i.e., $O_1H = \frac{R_1^2}{2R_2}$. By Problem 4.24 the surface area of the considered segment is equal to $\frac{2\pi R_2 \cdot R_1^2}{2R_2} = \pi R_1^2$.

4.31. If spheres α and β intersect, then the surface area of the part of sphere β situated inside sphere α constitutes $\frac{1}{4}$ of the surface area of α (Problem 4.30). Therefore, sphere β is contained inside α; hence, the ratio of their radii is equal to $\sqrt{5}$.

4.32. Let us consider a polyhedron circumscribed about sphere of radius 10; let the distance between any two points on the surface of this polyhedron not exceed 21 and let us prove that the number of the polyhedron's faces exceeds 20. First of all, observe that this polyhedron is situated inside the sphere of radius 11 whose center coincides with the center O of the inscribed sphere. Indeed, if for a point A from the surface of the polyhedron we have $OA > 11$, then let B be the other intersection point of the polyhedron's surface with line OA. Then

$$AB = AO + OB > 11 + 10 = 21$$

which is impossible.

For each face, its plane cuts off the sphere of radius 11 a "hat" of area $2\pi R(R-r)$, where $R = 11$ and $r = 10$ (see Problem 4.24). Such "hats" cover the whole sphere and, therefore, $n \cdot 2\pi R(R-r) \geq 4\pi R^2$, where n is the number of faces. Hence, $n \geq \frac{2R}{R-r} = 22 > 20$.

4.33. The planes of the cube's faces divide the circumscribed sphere into 12 "bilaterals" (corresponding to the edges of the cube) and 6 curvilinear quadrilaterals (corresponding to the faces of the cube). Let x be the area of the "bilateral", y the area of the "quadrilateral". Since the radius of the circumscribed sphere is equal to $\frac{a\sqrt{3}}{2}$, the plane of the cube's face cuts from the sphere a segment of height $\frac{a(\sqrt{3}-1)}{2}$. The surface area of this segment is equal to $\frac{1}{2}\pi a^2(3-\sqrt{3})$. This segment consists of four "bilaterals" and one "quadrilateral", i.e.,

$$4x + y = \frac{1}{2}\pi a^2(3-\sqrt{3}).$$

It is also clear that

$$12x + 6y = 4\pi R^2 = 3\pi a^2.$$

Solving the system of equations, we get

$$x = \frac{\pi a^2(2-\sqrt{3})}{4}; \quad y = \frac{\pi a^2(\sqrt{3}-1)}{2}.$$

4.34. Let us consider a regular octahedron with edge $2R$. The radius of the ball tangent to all its edges is equal to R. The faces of the octahedron divide the ball into 8 spherical segments (corresponding to faces) and 6 curvilinear quadrilaterals (corresponding to vertices). Let x be the area of a segment and y the area of a "quadrilateral". The areas to be found are equal to y and $5y + 4x$.

First, let us compute x. Since the distance from the center of octahedron to a vertex is equal to $\sqrt{2}R$ and the distance from the center of the octahedron's face to a vertex is equal to $\frac{2R}{\sqrt{3}}$, it follows that the distance from the center of octahedron to its face is equal to $R\sqrt{\frac{2}{3}}$. Therefore, the height of the considered spherical segment is equal to $(1-\sqrt{\frac{2}{3}})R$ and $x = 2\pi R^2(1-\sqrt{\frac{2}{3}})$. It is also clear that $8x + 6y = 4\pi R^2$. Therefore, $y = \frac{2\pi R^2}{3} \cdot \left(4\sqrt{\frac{2}{3}} - 3\right)$ and $5y + 4x = \pi R^2\left(\frac{16}{3}\sqrt{\frac{2}{3}} - 2\right)$.

4.35. Let us consider a right tetrahedron with edge 2. The surface of the sphere tangent to all its edges is divided by the tetrahedron's surface into 4 equal curvilinear triangles the area of each of which is the desired quantity and 4 equal segments. Let x be the distance from the center of a face to a vertex, y the distance from the center of the tetrahedron to a face, and z the distance from the center of a face to an edge of this face. It is easy to verify that $x = \frac{2}{\sqrt{3}}$ and $z = \frac{1}{\sqrt{3}}$. Further $y = \frac{h}{4}$, where $h = \sqrt{4 - x^2} = \sqrt{\frac{8}{3}}$ is the height of the tetrahedron, i.e., $y = \frac{1}{\sqrt{6}}$. The radius r of the sphere is equal to

$$\sqrt{y^2 + z^2} = \sqrt{\frac{1}{6} + \frac{1}{3}} = \frac{1}{\sqrt{2}}.$$

The height of each of the four segments is equal to $r - y = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}}$. Therefore, the area in question is equal to

$$\frac{1}{4}\left(4\pi\left(\frac{1}{\sqrt{2}}\right)^2 - 4 \cdot 2\pi\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}}\right)\right) = \pi\left(\frac{1}{\sqrt{3}} - \frac{1}{2}\right).$$

4.36. Let us consider a cube with edge $2\sqrt{2}$. A sphere of radius 2 whose center coincides with that of the cube is tangent to all its edges and its intersections with the faces are circles of radius $\sqrt{2}$. The surface of the sphere is divided by the surface of the cube into 6 spherical segments and 8 curvilinear triangles. Let x be the area of a spherical segment and y the area of a curvilinear triangle. Then the areas in question are equal to y and $16\pi - y - 3x$, respectively, where 16π is the surface area of the sphere of radius 2. Since the height of each spherical segment is equal to $2 - \sqrt{2}$, it follows that $x = 4\pi(2 - \sqrt{2})$, consequently, $y = \frac{16\pi - 6x}{8} = \pi(3\sqrt{2} - 4)$ and $16\pi - y - 3x = \pi(9\sqrt{2} - 4)$, respectively.

4.37. Let us introduce a coordinate system with the origin at the center of the first sphere and Ox-axis passing through the center of the second sphere. Let the distance between the centers of spheres be equal to a; the radii of the first and the second spheres be equal to R and r. Then the degrees of point (x, y, z) relative to the first and second spheres are equal to $x^2 + y^2 + z^2 - R^2$ and $(x - a)^2 + y^2 + z^2 - r^2$. Hence, the locus to be found is given by the equation

$$x^2 + y^2 + z^2 - R^2 = (x - a)^2 + y^2 + z^2 - r^2,$$

i.e., $x = \frac{a^2 + R^2 - r^2}{2a}$. This equation determines a plane perpendicular to the line that connects the sphere's centers.

4.38. Let M be the midpoint of segment AB; let l be the line that passes through the centers of given spheres; P the intersection point of line l and the radical plane of the given spheres. Since the tangents MA and MB drawn from point M to the given spheres are equal, it follows that M belongs to the radical plane of these spheres. Hence, the projection of point M to line l is point P, i.e., the projections of points A and B to line l are symmetric through P. Therefore, under the symmetry through P the projection of segment AC to line l turns into the projection of segment BD.

4.39. The midpoints of the common tangents to the two spheres lie in their radical plane. Let O_1 and O_2 be the centers of given spheres, M the midpoint of a common tangent, N the intersection point of the radical plane with line O_1O_2. Let us consider the section of given spheres by planes passing through points O_1

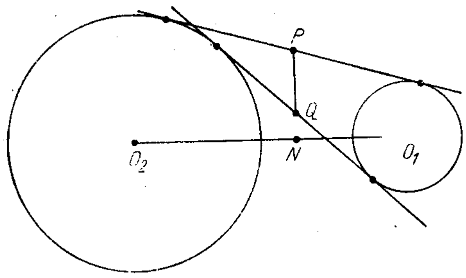

FIGURE 37 (SOL. 4.39)

and O_2 and draw outer and inner tangents to the circles obtained in the section (Fig. 37). Let P and Q be the midpoints of these tangents. Let us prove that $NQ \leq NM \leq NP$. Indeed,

$$NM^2 = O_1M^2 - O_1N^2 = \frac{x^2}{4} + R_1^2 - O_1N^2,$$

where x is the length of the tangent and x takes its greatest and least values in the cases of the inner and outer tangency accordingly (see the solution of Problem 4.4). Thus the locus to be found is the annulus situated in the radical plane; the outer radius of the annulus is NP and the inner one is NQ.

4.40. Let $S_1, \dots, S_n$ be the surfaces of the given balls. For every sphere S_i consider figure M_i that consists of points whose degree with respect to S_i does not exceed the degrees relative to all the other spheres. Let us prove that figure M_i is a convex one. Indeed, let M_{ij} be the figure consisting of points whose degree relative to S_i does not exceed the degree relative to S_j; figure M_{ij} is a half space consisting of the points that lie on the same side of the radical plane of spheres S_i and S_j as the sphere S_i. Figure M_i is the intersection of convex figures M_{ij}; hence, is convex itself. Moreover, M_i contains sphere S_i because each figure M_{ij} contains sphere S_i. For any point in space some of its degrees relative to spheres $S_1, \dots, S_n$ is the least one and, therefore, figures M_i cover the whole space. By considering the parts of these figures that lie inside the initial polyhedron we get the desired partition.

4.41. Let A be the intersection point of the given circles and O the vertex of the considered cone (or OA is the generator of the cylinder). Since line OA is perpendicular to the tangent to circle S_1 at point A, then circles S_1 and S_2 are perpendicular if and only if OA is tangent to S_2.

4.42. First, let us consider the spherical "bilateral" — the part of the sphere confined inside the dihedral angle of value α whose edge passes through the center of the sphere. The area of such a figure is proportional to α and for $\alpha = \pi$ it is equal to $2\pi R^2$; hence, it is equal to $2\alpha R^2$.

For the given trihedral angle, to every pair of the planes of the faces two "bilaterals" correspond. These "bilaterals" cover the given curvilinear triangle and the triangle symmetric to it through the center of the sphere in 3 coats; they cover the remaining part of the sphere in one coat. Hence, the sum of their areas is equal to the surface area of the sphere multiplied by $4S$, where S is the area of the triangle in question. Hence,

$$S = R^2(\alpha + \beta + \gamma - \pi).$$

4.43. Let us consider the set of endpoints of the arcs with the beginning at point C; let these arcs be divided in halves by the great circle passing through points A_1 and B_1. This set is the circle passing through points A, B and point C' symmetric to point C through the radius that divides arc $\smile A_1B_1$ in halves. A part of this circle consisting of the endpoints of the arcs that intersect side A_1B_1 of the curvilinear triangle A_1B_1C lies inside the curvilinear triangle ABC. In particular, inside this triangle lies point C'; hence,

$$S_{ABC} > S_{A_1B_1C} + S_{A_1B_1C'}.$$

We compare the areas of the curvilinear triangles. It remains to observe that $S_{A_1B_1C} = S_{A_1B_1C'}$, because the corresponding triangles are equal.

4.44. Let us cut the n-hedral angle into $n-2$ trihedral angles by drawing a plane through one of its edges and edges not adjacent to it. For each of these trihedral angles write the formula from Problem 4.42 and sum the formulas; we get the desired statement.

4.45. Let M and N be the intersection points of the sphere with the line passing through the center of circle S circumscribed about triangle ABC and perpendicular to its plane. Let $\alpha = \angle MBC = \angle MCB$, $\beta = \angle MAC = \angle MCA$ and $\gamma = \angle MAB = \angle MBA$ (we are talking about the spherical angles).

We can ascribe signs to these values in order to have $\beta + \gamma = \angle A$, $\alpha + \gamma = \angle B$ and $\alpha + \beta = \angle C$. Therefore, $2\gamma = \angle A + \angle B - \angle C$. Each of the angles $\angle A$, $\angle B$ and $\angle C$ is determined up to 2π; hence, the angle γ is determined up to π. The equality $\gamma = \angle MAB = \angle MBA$ determines two points M symmetric through the plane OAB, where O is the center of the sphere. If instead of γ we take $\gamma + \pi$, then instead of M we get point N, i.e., circle S does not vary. To the locus to be found not all the points of the circle's belong but only one of the arcs determined by points A and B; which exactly arc is clear by looking at the sign of the number $\angle A + \angle B - \angle C$. Thus, the locus consists of two arcs of the circles symmetric through plane OAB.

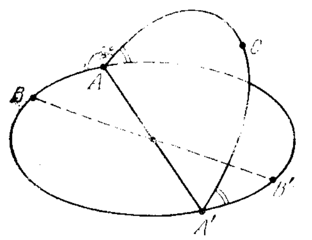

FIGURE 38 (SOL. 4.46)

4.46. The area of spherical triangle ABC is determined by the value $\angle A + \angle B + \angle C$ (see Problem 4.42). Let points A' and B' be diametrially opposite to points A and B. The angles of spherical triangles ABC and $A'B'C'$ are related as follows (see Fig. 38): $\angle A' = \pi - \angle A$, $\angle B' = \pi - \angle B$ and the angles at vertex C are equal. Hence,

$$\angle A' + \angle B' - \angle C = 2\pi - (\angle A + \angle B + \angle C)$$

is constant. The desired locus consists of two arcs of the circles passing through points A' and B' (cf. Problem 4.45).

4.47. Suppose that given arcs a, b and c do not intersect. Let C_a and C_b be intersection points of great circles containing arcs a and b. Since arc a is greater than $180°$, it contains one of these points, for example C_a. Then arc b contains point C_b. Let us also consider the intersection points A_b and A_c, B_a and B_c of the other pairs of great circles (A_b belongs to arc b, A_c to arc c, B_a to arc a and B_c to arc c). Points B_c and C_b lie in the plane of arc a but do not belong to arc a itself. Hence, $\angle B_cOC_b < 60°$, where O is the center of the sphere. Similarly, $\angle A_cOC_a < 60°$ and $A_bOB_a < 60°$. Therefore, $\angle A_cOB_c = \angle A_bOB_a < 60°$ and $A_cOC_b = 180° - \angle A_cOC_a > 120°$, i.e., $\angle A_cOB_c + \angle B_cOC_b < \angle A_cOC_b$. Contradiction.

4.48. Let O be the center of the sphere. To every plane passing through O we may assign a pair of points of the sphere — the intersection points with the sphere of the perpendicular to this plane passing through O. It is easy to verify that under this map to planes passing through point A the points of the great circle perpendicular to line OA correspond. Hence, to the planes that intersect arc $\smile AB$ there correspond the points from the part of the sphere confined between the two planes passing through point O perpendicularly to lines OA and OB, respectively (Fig. 39).

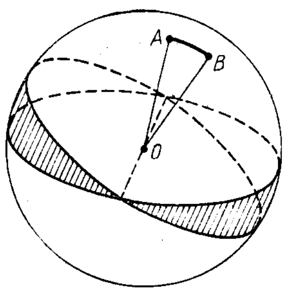

FIGURE 39 (SOL. 4.48)

The area of this figure is equal to $(\frac{\alpha}{\pi})S$, where α is the angle value of arc $\smile AB$ and S is the area of the sphere. Therefore, if the sum of the angle values of the arcs is smaller than π, then the area of the figure consisting of the points of the sphere corresponding to the planes that intersect these arcs is smaller than S.

4.49. a) The solid angle is proportional to the value of the dihedral angle and the solid angle of the trihedral angle of value π is equal to 2π.

b) See Problem 4.44.

4.50. Let O be the vertex of the cone and OH its height. Let us construct a sphere of radius 1 centered at O and consider its section by the plane passing through line OH. Let A and B be the points of the cone that lie on the sphere; M the intersection point of ray OH with the sphere (Fig. 40). Then $HM = OM - OH = 1 - \cos\alpha$. The solid angle of the cone is equal to the surface of the spherical segment cut by the base of the cone. By Problem 4.24 this area is equal to $2\pi Rh = 2\pi(1 - \cos\alpha)$.

4.51. The solid angle of the trihedral angle is equal to the sum of its dihedral angles minus π (see Problem 4.42) and, therefore, the sum of the solid angles of

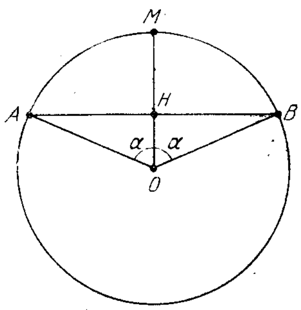

FIGURE 40 (SOL. 4.50)

the trihedral angles of the tetrahedron is equal to the doubled sum of its dihedral angles minus 4π. The doubled sum of the dihedral angles of the tetrahedron is equal to the sum of their solid angles.

4.52. The solid angle at the i-th vertex of the polyhedron is equal to $\sigma_i-(n_i-2)\pi$, where σ_i is the sum of the dihedral angles at the edges that go out of the vertex and n_i is the number of these edges (cf. Problem 4.44). Since each edge goes out exactly from two vertices, it follows that $\sum n_i = 2E$, where E is the number of edges. Therefore, the sum of the solid angles of the polyhedral angles is equal to $2\sigma - 2(E-V)\pi$, where σ is the sum of dihedral angles and V is the number of vertices. It remains to notice that $E - V = F - 2$ (Problem 8.14).

CHAPTER 5. TRIHEDRAL AND POLYHEDRAL ANGLES CHEVA'S AND MENELAUS'S THEOREMS FOR TRIHEDRAL ANGLES

§1. The polar trihedral angle

5.1. Given a trihedral angle with plane angles α, β, γ and the dihedral angles A, B and C, respectively, opposite to them, prove that there exists a trihedral angle with plane angles $\pi - A$, $\pi - B$ and $\pi - C$ and dihedral angles $\pi - \alpha$, $\pi - \beta$ and $\pi - \gamma$.

5.2. Prove that if dihedral angles of a trihedral angle are right ones, then its plane angles are also right ones.

5.3. Prove that trihedral angles are equal if the corresponding dihedral angles are equal.

§2. Inequalities with trihedral angles

5.4. Prove that the sum of two plane angles of a trihedral angle is greater than the third plane angle.

5.5. Prove that the sum of plane angles of a trihedral angle is smaller than 2π and the sum of its dihedral angles is greater than π.

5.6. A ray SC' lies inside the trihedral angle $SABC$ with vertex S. Prove that the sum of plane angles of a trihedral angle $SABC$ is greater than the sum of plane angles of the trihedral angle $SABC'$.

§3. Laws of sines and cosines for trihedral angles

5.7. Let α, β and γ be plane angles of a trihedral angle, A, B and C the dihedral angles opposite to them. Prove that

(The law of sines for a trihedral angle) $\sin\alpha : \sin A = \sin\beta : \sin B = \sin\gamma : \sin C.$

5.8. Let α, β and γ be plane angles of a trihedral angle A, B and C the dihedral angles opposite to them.

a) Prove that
(The first law of cosines for a trihedral angle)

$$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos A.$$

b) Prove that
(The second law of cosines for a trihedral angle)

$$\cos A = -\cos B\cos C + \sin B\sin C\cos\alpha.$$

5.9. Plane angles of a trihedral angle are equal to α, β and γ; the edges opposite to them form angles a, b and c with the planes of the faces. Prove that

$$\sin\alpha\sin a = \sin\beta\sin b = \sin\gamma\sin c.$$

5.10. a) Prove that if all the plane angles of a trihedral angle are obtuse ones, then all its dihedral angles are also obtuse ones.

b) Prove that if all the dihedral angles of a trihedral angle are acute ones, then all its plane angles are also acute ones.

§4. Miscellaneous problems

5.11. Prove that in an arbitrary trihedral angle the bisectors of two plane angles and the angle adjacent to the third plane angle lie in one plane.

5.12. Prove that the pairwise angles between the bisectors of plane angles of a trihedral angle are either simultaneously acute, or simultaneously obtuse, or simultaneously right ones.

5.13. a) A sphere tangent to faces SBC, SCA and SAB at points A_1, B_1 and C_1 is inscribed in trihedral angle $SABC$. Express the value of the angle ASB_1 in terms of the plane angles of the given trihedral angle.

b) The inscribed and escribed spheres of tetrahedron $ABCD$ are tangent to the face ABC at points P and P', respectively. Prove that lines AP and AP' are symmetric through the bisector of angle BAC.

5.14. The plane angles of a trihedral angle are not right ones. Through the vertices of tetrahedral angle planes perpendicular to the opposite faces are drawn. Prove that these planes intersect along one line.

5.15. a) The plane angles of a trihedral angle are not right ones. In the planes of the trihedral angle's faces there are drawn lines perpendicular to the respective opposite edges. Prove that all three lines are parallel to one plane.

b) Two trihedral angles with common vertex S are placed so that the edges of the second angle lie in the planes of the corresponding faces of the first angle and are perpendicular to its opposite edges. Find the plane angles of the first trihedral angle.

§5. Polyhedral angles

5.16. a) Prove that for any convex tetrahedral angle there exists a section which is a parallelogram and all such sections are parallel to each other.

b) Prove that there exists a section of a convex four-hedral angle with equal plane angles which is a rhombus.

5.17. Prove that any plane angle of a polyhedral angle is smaller than the sum of all the other plane angles.

5.18. One of two convex polyhedral angles with common vertex lies inside the other one. Prove that the sum of the plane angles of the inner polyhedral angle is smaller than the sum of the plane angles of the outer polyhedral angle.

5.19. a) Prove that the sum of dihedral angles of a convex n-hedral angle is greater than $(n-2)\pi$.

b) Prove that the sum of plane angles of a convex n-hedral angle is smaller than 2π.

5.20. The sum of plane angles of a convex n-hedral angle is equal to the sum of its dihedral angles. Prove that $n = 3$.

5.21. A sphere is inscribed in a convex four-hedral angle. Prove that the sums of its opposite plane angles are equal.

5.22. Prove that a convex four-hedral angle can be inscribed in a cone if and only if the sums of its opposite dihedral angles are equal.

§6. Ceva's and Menelaus's theorems for trihedral angles

Before we pass to Ceva's and Menelaus's theorems for trihedral angles we have to prove (and formulate) Ceva's and Menelaus's theorems for triangles. To formulate these theorems, we need the notion of the ratio of oriented segments that lie on the same line.

Let points A, B, C and D lie on one line. By the *ratio of oriented segments* AB and CD we mean the number $\frac{\overline{AB}}{\overline{CD}}$ whose absolute value is equal to $\frac{AB}{CD}$ and which is positive if vectors $\{AB\}$ and $\{CD\}$ are similarly directed and negative if the directions of these vectors are opposite.

5.23. On sides AB, BC and CA of triangle ABC (or on their extensions), points C_1, A_1 and B_1,respectively, are taken.

a) Prove that points A_1, B_1 and C_1 lie on one line if and only if

(Menelaus's theorem) $$\frac{\overline{A_1B}}{\overline{A_1C}} \cdot \frac{\overline{B_1C}}{\overline{B_1A}} \cdot \frac{\overline{C_1A}}{\overline{C_1B}} = 1.$$

b) Prove that if lines AA_1, BB_1 and CC_1 are not pairwise parallel, then they intersect at one point if and only if

(Ceva's theorem) $$\frac{\overline{A_1B}}{\overline{A_1C}} \cdot \frac{\overline{B_1C}}{\overline{B_1A}} \cdot \frac{\overline{C_1A}}{\overline{C_1B}} = -1.$$

Let rays l, m and n with a common origin lie in one plane. In this plane, select a positive direction of rotation. In this section we will denote by $\frac{\sin(l,m)}{\sin(n,m)}$ the ratio of sines of the angles through which one has to rotate in the positive direction rays l and n in order for them to coincide with ray m. Clearly, this ratio does not depend on the choice of the positive direction of the rotation in plane: as we vary this direction both the numerator and the denominator adjust accordingly.

Let half-planes α, β and γ have a common boundary. Select one of the positive directions of rotation about this line (the boundary) as the positive one. In this section we will denote by $\frac{\sin(\alpha,\beta)}{\sin(\gamma,\beta)}$ the ratio of the sines of the angles through which one has to turn in the positive direction the half-planes α and γ in order for them to coincide with β. Clearly, this quantity does not depend on the choice of the positive direction of rotation.

5.24. Given a trihedral angle with vertex S and edges a, b and c. Rays α, β and γ starting from S lie in the planes of the faces opposite to edges a, b and c, respectively.

a) Prove that rays α, β and γ lie in one plane if and only if

(First Menelaus's theorem) $$\frac{\sin(a,\gamma)}{\sin(b,\gamma)}\cdot\frac{\sin(b,\alpha)}{\sin(c,\alpha)}\cdot\frac{\sin(c,\beta)}{\sin(a,\beta)}=1.$$

b) Prove that planes passing through pairs of rays a and α, b and β, c and γ intersect along one line if and only if

(First Ceva's theorem) $$\frac{\sin(a,\gamma)}{\sin(b,\gamma)}\cdot\frac{\sin(b,\alpha)}{\sin(c,\alpha)}\cdot\frac{\sin(c,\beta)}{\sin(a,\beta)}=-1.$$

5.25. Given are a trihedral angle with vertex S and edges a, b, c and rays α, β and γ, respectivly, starting from S and lying in the planes of the faces opposite to these edges. Let l and m be two rays with a common vertex. Denote by lm the plane determined by these rays.

a) Prove that

$$\frac{\sin(ab,a\alpha)}{\sin(ac,a\alpha)}\cdot\frac{\sin(bc,b\beta)}{\sin(ba,b\beta)}\cdot\frac{\sin(ca,c\gamma)}{\sin(cb,c\gamma)}=\frac{\sin(b,\alpha)}{\sin(c,\alpha)}\cdot\frac{\sin(c,\beta)}{\sin(a,\beta)}\cdot\frac{\sin(a,\gamma)}{\sin(b,\gamma)}.$$

b) Prove that rays α, β and γ lie in one plane if and only if

(Second Menelaus's theorem) $$\frac{\sin(ab,a\alpha)}{\sin(ac,a\alpha)}\cdot\frac{\sin(bc,b\beta)}{\sin(ba,b\beta)}\cdot\frac{\sin(ca,c\gamma)}{\sin(cb,c\gamma)}=1.$$

c) Prove that the planes passing through pairs of rays a and α, b and β, c and γ intersect along one line if and only if

(Second Ceva's theorem) $$\frac{\sin(ab,a\alpha)}{\sin(ac,a\alpha)}\cdot\frac{\sin(bc,b\beta)}{\sin(ba,b\beta)}\cdot\frac{\sin(ca,c\gamma)}{\sin(cb,c\gamma)}=-1.$$

5.26. In trihedral angle $SABC$, a sphere tangent to faces SBC, SCA and SAB at points A_1, B_1 and C_1, respectively, is inscribed. Prove that planes SAA_1, SBB_1 and SCC_1 intersect along one line.

5.27. Given a trihedral angle with vertex S and edges a, b and c. R are placed in planes of the faces opposite to edges a, b and c. Let rays α', β' and γ' be symmetric to rays α, β and γ, respectively, through the bisectors of the corresponding faces.

a) Prove that rays α, β and γ lie in one plane if and only if rays α', β' and γ' lie in one plane.

b) Prove that the planes passing through pairs of rays a and α, b and β, c and γ intersect along one line if and only if the planes passing through the pairs of rays a and α', b and β', c and γ' intersect along one line.

5.28. Given a trihedral angle with vertex S and edges a, b and c. Lines α, β and γ lie in the planes of the faces opposite to edges a, b and c, respectively. Let α' be the line along which the plane symmetric to the plane $a\alpha$ through the bisector plane of the dihedral angle at edge a intersects the plane of face bc; lines β' and γ' are similarly defined.

a) Prove that lines α, β and γ lie in one plane if and only if the lines α', β' and γ' lie in one plane.

b) Prove that the planes passing through pairs of lines a and α, b and β, c and γ intersect along one line if and only if the planes passing through the pairs of lines a and α', b and β', c and γ' intersect along one line.

5.29. Given tetrahedron $A_1A_2A_3A_4$ and a point P. For every edge A_iA_j consider the plane symmetric to plane PA_iA_j through the bisector plane of the dihedral angle at edge A_iA_j. Prove that either all these 6 planes intersect at one point or all of them are parallel to one line.

5.30. Given trihedral angle $SABC$ such that $\angle ASB = \angle ASC = 90^\circ$. Planes π_b and π_c pass through edges SB and SC and planes π_b' and π_c' are symmetric to π_b and π_c, respectively, through the bisector planes of the dihedral angles at these edges. Prove that the projections of the intersection lines of planes π_b and π_c, π_b' and π_c' to plane BSC are symmetric through the bisector of angle $\angle BSC$.

5.31. Let the Monge's point of tetrahedron $ABCD$ (see Problem 7.32) lie in the plane of face ABC. Prove that through point D planes pass in which there lie:

a) intersection points of the heights of faces DAB, DBC and DAC;

b) the centers of the circumscribed circles of faces DAB, DBC and DAC.

Problems for independent study

5.32. A sphere with center O is inscribed in the trihedral angle with vertex S. Prove that the plane passing through the three tangent points is perpendicular to line OS.

5.33. Given trihedral angle $SABC$ with vertex S; the dihedral angles $\angle A$, $\angle B$ and $\angle C$ at edges SA, SB and SC; the plane angles α, β and γ opposite to them.

a) The bisector plane of the dihedral angle at edge SA intersects face SBC along ray SA_1. Prove that

$$\sin A_1SB : \sin A_1SC = \sin ASB : \sin ASC.$$

b) The plane passing through edge SA perpendicularly to face SBC intersects this face along ray SA_1. Prove that

$$\sin A_1SB : \sin A_1SC = (\sin\beta\cos C) : (\sin\gamma\cos B).$$

We assume here that all the plane angles of the given trihedral angle are acute ones; consider on your own the case when among the plane angles of the trihedral angle obtuse angles are encountered.

5.34. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the unit vectors directed along the edges of trihedral angle $SABC$.

a) Prove that the planes passing through the edges of the trihedral angle and the bisectors of the opposite faces intersect along one line and this line is given by vector $\mathbf{a}+\mathbf{b}+\mathbf{c}$.

b) Prove that the bisector planes of the dihedral angles of the trihedral angle intersect along one line and this line is given by the vector

$$\mathbf{a}\sin\alpha+\mathbf{b}\sin\beta+\mathbf{c}\sin\gamma.$$

c) Prove that the planes passing through the edges of the trihedral angle perpendicularly to their opposite faces intersect along one line and this line is given by the vector

$$\mathbf{a}\sin\alpha\cos B\cos C+\mathbf{b}\sin\beta\cos A\cos C+\mathbf{c}\sin\gamma\cos A\cos B.$$

d) Prove that the planes passing through the bisectors of the faces perpendicularly to the planes of these faces intersect along one line and this line is determined by the vector

$$[\mathbf{a},\mathbf{b}]+[\mathbf{b},\mathbf{c}]+[\mathbf{c},\mathbf{a}]$$

(Recall the definition of the *vector product* $[\mathbf{a},\mathbf{b}]$ of vectors $\mathbf{a}$ and $\mathbf{b}$.)

5.35. In a convex tetrahedral angle the sums of the opposite plane angles are equal. Prove that a sphere can be inscribed in this tetrahedral angle.

5.36. Projections SA', SB' and SC' of edges SA, SB and SC of a trihedral angle to the faces opposite to them form the edges of a new trihedral angle. Prove that the bisector planes of the new angle are SAA', SBB' and SCC'.

Solutions

5.1. Inside the given trihedral angle with vertex S take an arbitrary point S' and from it drop perpendiculars $S'A'$, $S'B'$ and $S'C'$ to faces SBC, SAC and SAB, respectively. Clearly, the plane angles of trihedral angle $S'A'B'C'$ complement the dihedral angles of trihedral angle $SABC$ to π. To complete the proof it remains to notice that edges SA, SB and SC are perpendicular to faces $S'B'C'$, $S'A'C'$ and $S'A'B'$, respectively.

Angle $S'A'B'C'$ is called the *complementary* or *polar* one to angle $SABC$.

5.2. Consider the trihedral angle polar to the given one (see Problem 5.1). Its plane angles are right ones; hence, its dihedral angles are also right ones. Therefore, the plane angles of the initial trihedral angle are also right ones.

5.3. The angles polar to the given trihedral angles have equal plane angles; hence, they are equal themselves.

5.4. Consider trihedral angle $SABC$ with vertex S. The inequality $\angle ASC < \angle ASB + \angle BSC$ is obvious if $\angle ASC \leq \angle ASB$. Therefore, let us assume that $\angle ASC \geq \angle ASB$. Then, inside face ASC, we can select a point B' so that $\angle ASB' = \angle ASB$ and $SB' = SB$, i.e., $\angle ASB = \angle ASB'$. We may assume that point C lies in plane ABB'. Since

$$AB'+B'C = AC < AB+BC = A'B+BC,$$

it follows that $B'C < BC$. Hence, $\angle B'SC < \angle BSC$. It remains to notice that $\angle B'SC = \angle ASC - \angle ASB$.

5.5. First solution. On the edges of the trihedral angle draw equal segments SA, SB and SC starting from vertex S. Let O be the projection of S to plane ABC. The isosceles triangles ASB and AOB have a common base AB and $AS > AO$. Hence, $\angle ASB < \angle AOB$. By writing similar inequalities for the two other angles and taking their sum we get

$$\angle ASB + \angle BSC + \angle CSA < \angle AOB + \angle BOC + \angle COA \leq 2\pi.$$

The latter inequality becomes a strict one only if point O lies outside triangle ABC.

To prove the second part, it suffices to apply the already proved inequality to the angle polar to the given one (see Problem 5.1). Indeed, if α, β and γ are dihedral angles of the given trihedral angle, then

$$(\pi - \alpha) + (\pi - \beta)(\pi - \gamma) < 2\pi,$$

i.e., $\alpha + \beta + \gamma > \pi$.

Second solution. Let point A' lie on the extension of edge SA beyond vertex S. By Problem 5.4

$$\angle A'SB + \angle A'SC > \angle BSC, \text{ i.e., } (\pi - \angle ASB) + (\pi - \angle ASC) > \angle BSC;$$

hence, $2\pi > \angle ASB + \angle BSC + \angle CSA$.

Proof of the second part of the problem is performed as in the first solution.

5.6. Let K be the intersection point of face SCB with line AC'. By Problem 5.4 we have $\angle C'SK + \angle KSB > \angle C'SB$ and

$$\angle CSA + \angle CSK > \angle ASK = \angle ASC' + \angle C'SK.$$

Adding these inequalities and taking into account that $\angle CSK + \angle KSB = \angle CSB$ we get the desired statement.

5.7. On edge SA of trihedral angle $SABC$, take an arbitrary point M. Let M' be the projection of M to plane SBC, let P and Q be the projections of M to lines SB and SC. By the theorem on three perpendiculars $M'P \perp SB$ and $M'Q \perp SC$. If $SM = a$, then $MQ = a \sin \beta$ and

$$MM' = MQ \sin C = a \sin \beta \sin C.$$

Similarly,

$$MM' = MP \sin B = a \sin \gamma \sin B.$$

Therefore,

$$\sin \beta : \sin B = \sin \gamma : \sin C.$$

The second equality is similarly proved.

5.8. a) **First solution.** On segment SA take a point, M, and at it erect perpendiculars PM and QM to edge SA in planes SAB and SAC, respectively (points P and Q lie on lines SB and SC). By expressing the length of the side PQ in triangles PQM and PQS with the help of the law of cosines and equating these expressions we get the desired equality after simplifications.

Second solution. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be unit vectors directed along edges SA, SB and SC, respectively. Vector $\mathbf{b}$ lying in plane SAB can be represented in the form

$$\mathbf{b} = \mathbf{a}\cos\gamma + \mathbf{u}, \text{ where } \mathbf{u} \perp \mathbf{a} \text{ and } |\mathbf{u}| = \sin\gamma.$$

Similarly,

$$\mathbf{c} = \mathbf{a}\cos\beta + \mathbf{v}, \text{ where } \mathbf{v} \perp \mathbf{a} \text{ and } |\mathbf{v}| = \sin\beta.$$

It is also clear that the angle between vectors $\mathbf{u}$ and $\mathbf{v}$ is equal to $\angle A$.

On the one hand, the inner product of vectors $\mathbf{b}$ and $\mathbf{c}$ is equal to $\cos\alpha$. On the other hand, the product is equal to

$$(\mathbf{a}\cos\gamma + \mathbf{u},\ \mathbf{a}\cos\beta + \mathbf{v}) = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos\angle A.$$

b) To prove it, it suffices apply the first law of cosines to the angle polar to the given trihedral angle (cf. Problem 5.1).

5.9. Let us draw three planes parallel to the faces of the trihedral angle at distance 1 from them and intersecting the edges. Together with the planes of the faces they constitute a parallelepiped all the heights of which are equal to 1 and, therefore, the areas of all its faces are equal. Now, notice that the lengths of the edges of this parallelepiped are equal to $\frac{1}{\sin a}$, $\frac{1}{\sin b}$ and $\frac{1}{\sin c}$. Therefore, the areas of its faces are equal to

$$\frac{\sin\alpha}{\sin b\sin c}, \ \frac{\sin\beta}{\sin a\sin c}, \text{ and } \frac{\sin\gamma}{\sin a\sin b}.$$

By equating these expressions we get the desired statement.

5.10. a) By the first theorem on cosines for a trihedral angle (Problem 5.8 a))

$$\sin\beta\sin\gamma\cos A = \cos\alpha - \cos\beta\cos\gamma.$$

By the hypothesis $\cos\alpha < 0$ and $\cos\beta\cos\gamma > 0$; hence, $\cos A < 0$.

b) To prove it, it suffices to make use of the second theorem on cosines (Problem 5.8 b)).

5.11. First solution. On the edges of the trihedral angle, draw equal segments SA, SB and SC beginning from vertex S. The bisectors of angles ASB and BSC pass through the midpoints of segments AB and BC, respectively, and the bisector of the angle adjacent to angle CSA is parallel to CA.

Second solution. On the segments of the trihedral angle draw equal vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ beginning from vertex S. The bisectors of angles ASB and BSC are parallel to vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{b}+\mathbf{c}$ and the bisector of the angle adjacent to angle CSA is parallel to the vector $\mathbf{c}-\mathbf{a}$. It remains to notice that

$$(\mathbf{a}+\mathbf{b}) + (\mathbf{c}-\mathbf{a}) = \mathbf{b}+\mathbf{c}.$$

5.12. On the edges of the trihedral angle draw unit vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ starting from its vertex. Vectors $\mathbf{a}+\mathbf{b}$, $\mathbf{b}+\mathbf{c}$ and $\mathbf{a}+\mathbf{c}$ determine the bisectors of the plane angles. It remains to verify that all the pairwise inner products of these sums are of the same sign. It is easy to see that the inner product of any pair of these vectors is equal to

$$1 + (\mathbf{a},\mathbf{b}) + (\mathbf{b},\mathbf{c}) + (\mathbf{c},\mathbf{a}).$$

5.13. a) Let α, β and γ be the plane angles of trihedral angle $SABC$; let $x = \angle ASB_1 = \angle ASC_1$, $y = \angle BSA_1 = \angle BSC_1$ and $z = \angle CSA_1 = \angle CSB_1$. Then

$$x + y = \angle ASC_1 + \angle BSC_1 = \angle ASB = \gamma, \ \ y + z = \alpha, \ \ z + x = \beta.$$

Hence,

$$x = \frac{1}{2}(\beta + \gamma - \alpha).$$

b) Let point D' lie on the extension of edge AD beyond point A. Then the escribed sphere of the tetrahedron tangent to face ABC is inscribed in trihedral angle $ABCD'$ with vertex A. From the solution of heading a) it follows that

$$\angle BAP = \frac{\angle BAC + \angle BAD - \angle CAD}{2};$$

$$\angle CAP' = \frac{\angle BAC + \angle CAD' - \angle BAD'}{2}.$$

Since $\angle BAD' = 180^\circ - \angle BAD$ and $\angle CAD' = 180^\circ - \angle CAD$, we see that $\angle BAP = \angle CAP'$; hence, lines AP and AP' are symmetric through the bisector of angle BAC.

5.14. Let us select points A, B and C on the edges of the trihedral angle with vertex S so that $SA \perp ABC$ (the plane that passes through point A of one edge perpendicularly to the edge intersects the other two edges because the plane angles are not right ones). Let AA_1, BB_1 and CC_1 be the heights of triangle ABC. It suffices to verify that SAA_1, SBB_1 and SCC_1 are the planes spoken about in the formulation of the problem.

Since $BC \perp AS$ and $BC \perp AA_1$, it follows that $BC \perp SAA_1$; hence, planes SBC and SAA_1 are perpendicular to each other. Since $BB_1 \perp SA$ and $BB_1 \perp AS$, we see that $BB_1 \perp SAC$ and, therefore, planes SBB_1 and SAC are perpendicular. We similarly prove that planes SCC_1 and SBC are perpendicular to each other.

5.15. a) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be vectors directed along the edges SA, SB and SC of the trihedral angle. The line lying in plane SBC and perpendicular to edge SA is parallel to vector $(\mathbf{a}, \mathbf{b})\mathbf{c} - (\mathbf{a}, \mathbf{c})\mathbf{b}$. Similarly, two other lines are parallel to vectors $(\mathbf{b}, \mathbf{c})\mathbf{a} - (\mathbf{b}, \mathbf{a})\mathbf{c}$ and $(\mathbf{c}, \mathbf{a})\mathbf{b} - (\mathbf{c}, \mathbf{b})\mathbf{a}$. Since the sum of these vectors is equal to $\mathbf{0}$, they are parallel to one plane.

b) Let us direct vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ along the edges of the first trihedral angle $SABC$. Let $(\mathbf{b}, \mathbf{c}) = \alpha, (\mathbf{a}, \mathbf{c}) = \beta$ and $(\mathbf{a}, \mathbf{b}) = \gamma$. If the edge of the second angle, which lies in plane SAB, is parallel to vector $\lambda\mathbf{a} + \mu\mathbf{a}$, then $(\lambda\mathbf{a} + \mu\mathbf{b}, \mathbf{c}) = 0$, i.e., $\lambda\beta + \mu\alpha = 0$. It is easy to verify that if at least one of the numbers α and β is nonzero, then this edge is parallel to vector $\alpha\mathbf{a} - \beta\mathbf{b}$ (the case when one of these numbers is equal to zero should be considered separately).

Therefore, if not more than one of the numbers α, β and γ is equal to zero, then the edges of the second dihedral angle are parallel to vectors $\gamma\mathbf{c} - \beta\mathbf{b}$, $\alpha\mathbf{a} - \gamma\mathbf{c}$ and $\beta\mathbf{b} - \alpha\mathbf{a}$, and since the sum of these vectors is equal to zero, the edges should lie in one plane.

If, for example, $\alpha \neq 0$ and $\beta = \gamma = 0$, then two edges should be parallel to vector $\mathbf{a}$. There remains a unique possibility: all the numbers α, β and γ are equal to 0, i.e., the plane angles of the first trihedral angle are right ones.

5.16. a) Let A, B, C and D be (?)the points on the edges of a convex four-hedral angle with vertex S. Lines AB and CD are parallel if and only if they are parallel to line l_1 along which planes SAB and SCD intersect. Lines BC and AD are parallel if and only if they are parallel to line l_2 along which planes SCB and SAD intersect. Hence, the section is a parallelogram if and only if it is parallel to lines l_1 and l_2.

REMARK. For a non-convex four-hedral angle the section by the plane parallel to lines l_1 and l_2 is not a bounded figure.

b) Points A and C on the edges of a four-hedral angle can be selected so that $SA = SC$. Let P be the intersection point of segment AC with plane SBD. Points B and D can be selected so that $SB = SD$ and segment BD passes through point P. Since the plane angles of the given four-hedral angle are equal, the triangles SAB, SAD, SCB and SCD are equal. Therefore, quadrilateral $ABCD$ is a rhombus.

5.17. Consider a polyhedral angle $OA_1 \dots A_n$ with vertex O. As follows from the result of Problem 5.4

$$\angle A_1OA_2 < \angle A_2OA_3 + \angle A_1OA_3,\ \angle A_1OA_3 < \angle A_3OA_4 + \angle A_1OA_4, \dots$$
$$\dots,\quad \angle A_1OA_{n-1} < \angle A_{n-1}OA_n + \angle A_nOA_1.$$

Hence,

$$\angle A_1OA_2 < \angle A_2OA_3 + \angle A_3OA_4 + \dots + \angle A_{n-1}OA_n + \angle A_nOA_1.$$

5.18. Let polyhedral angle $OA_1 \dots A_n$ lie inside polyhedral angle $OB_1 \dots B_m$. We may assume that $A_1, \dots, A_n$ and $B_1, \dots, B_m$ are the intersection points of their edges with the unit sphere.

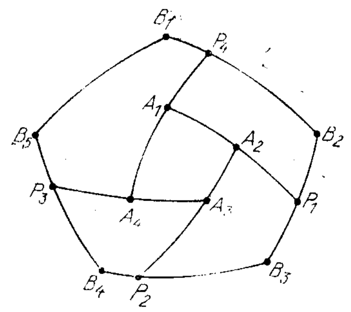

FIGURE 41 (SOL. 5.18)

Then the vertices of plane angles of the given polyhedral angles are equal to the lengths of the corresponding arcs of the sphere. Thus, instead of polyhedral angles, we will consider "spherical polygons" $A_1 \dots A_n$ and $B_1 \dots B_m$. Let $P_1, \dots, P_n$ be the points of intersection of "rays" $A_1A_2, \dots, A_nA_1$ with the sides of spherical polygon $B_1 \dots B_m$ (Fig. 41). By Problem 5.17

$$\smile A_iA_{i+1} + \smile A_{i+1}P_i = \smile A_iP_i < \smile A_iP_{i-1} + l(P_{i-1}, P_i),$$

where $l(P_{i-1}, P_i)$ is the length of the part of the "perimeter" of polygon $B_1 \dots B_m$ confined inside the "angle" $P_{i-1}A_iP_i$. By adding up these inequalities we get the desired statement.

5.19. a) Let us cut the n-hedral angle $SA_1 \dots A_n$ with vertex S into $n-2$ trihedral angles by planes SA_1A_3, SA_1A_4, $\dots$, SA_1A_{n-1}. The sum of dihedral angles of the n-hedral angle is equal to the sum of dihedral angles of these trihedral angles and the sum of dihedral angles of any trihedral angle is greater than π (Problem 5.5).

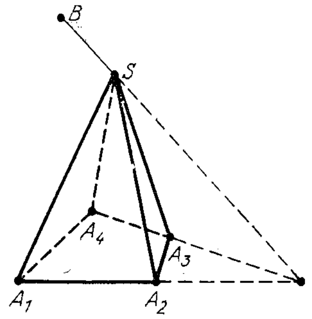

FIGURE 42 (SOL. 5.19)

b) Let us prove this statement by induction on n. For $n = 3$ it is true (cf. Problem 5.5). Suppose it is true for any convex $(n-1)$-hedral angle; let us prove then that it holds for a convex n-hedral angle $SA_1 \dots A_n$ with vertex S. Planes SA_1A_2 and $SA_{n-1}A_n$ have a common point, S, hence, they intersect along a line l which does not lie in plane SA_1A_n. On line l, take point B so that B and the polyhedral angle $SA_1 \dots A_n$ lie on different sides of the plane SA_1A_n (Fig. 42). Consider $(n-1)$-hedral angle $SBA_2A_3 \dots A_{n-1}$. By the inductive hypothesis the sum of its plane angles is smaller than 2π. By Problem 5.4

$$\angle BSA_1 + \angle BSA_n > \angle A_1SA_n.$$

Hence, the sum of the plane angles of the n-hedral angle $SA_1A_2 \dots A_n$ is smaller than the sum of the plane angles of the $(n-1)$-hedral angle $SBA_2A_3 \dots A_{n-1}$.

5.20. The sum of plane angles of an arbitrary convex polyhedral angle is smaller than 2π (see Problem 5.19 b)) and the sum of the dihedral angles of the convex n-hedral angle is greater than $(n-2)\pi$ (see Problem 5.19 a)). Hence, $(n-2)\pi < 2\pi$, i.e., $n < 4$.

5.21. Let the sphere be tangent to the faces of the tetrahedral angle $SABCD$ at points K, L, M and N, where K belongs to face SAB, L to face SBC, etc. Then

$$\angle ASK = \angle ASN, \quad \angle BSK = \angle BSL, \quad \angle CSL = \angle CSM, \quad \angle DSM = \angle DSN.$$

Therefore,

$$\begin{aligned}\angle ASD + \angle BSC = \angle ASN + \angle DSN + \angle BSL + \angle CSL = \\ \angle ASK + \angle DSM + \angle BSK + \angle CSM = \angle ASB + \angle CSD.\end{aligned}$$

5.22. Let the edges of the tetrahedral angle $SABCD$ with vertex S be generators of the cone with axis SO. In the trihedral angle formed by the rays SO, SA and SB; let the dihedral angles at edges SA and SB be equal. By considering three other such angles we deduce that the sums of the opposite dihedral angles of the tetrahedral angle $SABCD$ are equal.

Now, suppose that the sums of the opposite dihedral angles are equal. Let us consider the cone with generators SB, SA and SC. Suppose that SD is not its generator. Let SD_1 be the intersection line of the cone with plane ASD. In tetrahedral angles $SABCD$ and $SABCD_1$ the sums of the opposite dihedral angles are equal. It follows that the dihedral angles of trihedral angle $SCDD_1$ satisfy the relation $\angle D + \angle D_1 - 180^\circ = \angle C$.

Consider the trihedral angle polar to $SCDD_1$ (cf. Problem 5.1). In this angle the sum of two plane angles is equal to the third one; this is impossible thanks to Problem 5.4.

5.23. a) Let the projection to the line perpendicular to line A_1B_1 send points A, B and C to A', B' and C', respectively, and point C_1 to Q. Let both points A_1 and B_1 go into one point, P. Since

$$\frac{\overline{A_1B}}{\overline{A_1C}} = \frac{\overline{PB'}}{\overline{PC'}},\quad \frac{\overline{B_1C}}{\overline{B_1A}} = \frac{\overline{PC'}}{\overline{PA'}},\quad \frac{\overline{C_1A}}{\overline{C_1B}} = \frac{\overline{QA'}}{\overline{QB'}},$$

it follows that

$$\frac{\overline{A_1B}}{\overline{A_1C}} \cdot \frac{\overline{B_1C}}{\overline{B_1A}} \cdot \frac{\overline{C_1A}}{\overline{C_1B}} = \frac{\overline{PB'}}{\overline{PC'}} \cdot \frac{\overline{PC'}}{\overline{PA'}} \cdot \frac{\overline{QA'}}{\overline{QB'}} = \frac{\overline{PB'}}{\overline{PA'}} \cdot \frac{\overline{QA'}}{\overline{QB'}} = \frac{b}{a} \cdot \frac{a+x}{b+x}, \quad \text{where } |x| = PQ.$$

The equality $\frac{b}{a} \cdot \frac{a+x}{b+x} = 1$ is equivalent to the fact that $x = 0$ (we have to take into account that $a \neq b$ because $A' \neq B'$). But the equality $x = 0$ means that $P = Q$, i.e., point C_1 lies on line A_1B_1.

b) First, let us prove that if lines AA_1, BB_1 and CC_1 pass through one point, O, then the indicated relation holds. Let $\mathbf{a} = \{OA\}$, $\mathbf{b} = \{OB\}$ and $\mathbf{c} = \{OC\}$. Since point C_1 lies on line AB, it follows that

$$\{OC_1\} = \{OA\} + x\{AB\} = \mathbf{a} + x(\mathbf{b} - \mathbf{a}) = (1-x)\mathbf{a} + x\mathbf{b}.$$

On the other hand, point C_1 lies on line OC, therefore, $\{OC_1\} + \gamma\{OC\} = \{0\}$, i.e.,

$$(1-x)\mathbf{a} + x\mathbf{b} + \gamma\mathbf{c} = \mathbf{0}.$$

Similar arguments for points A_1 and B_1 show that

$$(1-y)\mathbf{b} + y\mathbf{c} + \alpha\mathbf{a} = \mathbf{0};\quad (1-z)\mathbf{c} + z\mathbf{a} + \beta\mathbf{b} = \mathbf{0}.$$

Since vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are pairwise noncolinear, all triples of nonzero numbers (p, q, r) for which

$$p\mathbf{a} + q\mathbf{b} + r\mathbf{c} = \mathbf{0}$$

are proportional. The comparison of the first and the third of the obtained equalities yield $\frac{1-x}{x} = \frac{z}{\beta}$ and the comparison of the second and the third ones yields $-\frac{1-y}{y} = \frac{\beta}{1-z}$. Consequently,

$$\frac{1-x}{x} \cdot \frac{1-y}{y} \cdot \frac{1-z}{z} = 1.$$

It remains to notice that

$$\frac{\overline{C_1B}}{\overline{C_1A}} = -\frac{1-x}{x}, \quad \frac{\overline{A_1C}}{\overline{A_1B}} = -\frac{1-y}{y}, \quad \frac{\overline{B_1A}}{\overline{B_1C}} = -\frac{1-z}{z}.$$

Now, suppose that the indicated relation holds and prove that then lines AA_1, BB_1 and CC_1 intersect at one point. Let C_1^* be the intersection point of line AB with the line passing through point C and the intersection point of lines AA_1 and BB_1. For point C_1^* the same relation holds as for point C_1. Therefore,

$$\frac{\overline{C_1^*A}}{\overline{C_1^*B}} = \frac{\overline{C_1A}}{\overline{C_1B}}.$$

Hence, $C_1^* = C_1$, i.e., lines AA_1, BB_1 and CC_1 meet at one point.

We can also verify that if the indicated relation holds and two of the lines AA_1, BB_1 and CC_1 are parallel, then the third line is also parallel to them.

5.24. a) On edges a, b and c of the trihedral angle, take arbitrary points A, B and C. Let A_1, B_1 and C_1 be points at which rays α, β and γ (or their continuations) intersect lines BC, CA and AB. By applying the law of sines to triangles SA_1B and SA_1C we get $\frac{A_1B}{\sin BSA_1} = \frac{BS}{\sin BA_1S}$ and $\frac{A_1C}{\sin CSA_1} = \frac{CS}{\sin CA_1S}$. Taking into account that $\sin BA_1S = \sin CA_1S$ we get $\frac{\sin BSA_1}{\sin CSA_1} = \frac{A_1B}{A_1C} \cdot \frac{CS}{BS}$. As is easy to verify, this means that

$$\frac{\sin(b,\alpha)}{\sin(c,\alpha)} = \frac{\overline{A_1B}}{\overline{A_1C}} \cdot \frac{CS}{BS}$$

(one only has to verify that the signs of these quantities coincide). Similarly, $\frac{\sin(a,\gamma)}{\sin(b,\gamma)} = \frac{\overline{C_1A}}{\overline{C_1B}} \cdot \frac{BS}{AS}$ and $\frac{\sin(c,\beta)}{\sin(a,\beta)} = \frac{\overline{B_1C}}{\overline{B_1A}} \cdot \frac{AS}{CS}$. It only remains to apply Menelaus's theorem to triangle ABC and notice that rays α, β and γ lie in one plane if and only if points A_1, B_1 and C_1 lie on one line.

The above solution has a small gap: we do not take into account the fact that the lines on which rays α, β and γ lie can be parallel to lines BC, CA and AB. In order to avoid this, points A, B and C should not be taken at random. Let A be an arbitrary point on edge a and P and Q be points on edges b and c, respectively, such that $AP \parallel \gamma$ and $AQ \parallel \beta$. On edge p, take point B distinct from P and let R be a point on edge c such that $BR \parallel \alpha$. It remains to take on edge c a point C distinct from Q and R. Now, points A_1, B_1 and C_1 at which the rays α, β and γ (or their extensions) intersect lines BC, CA and AB, respectively, always exist.

b) The solution almost literally repeats that of the preceding heading; one only has to apply to triangle ABC not Menelaus's theorem but Ceva's theorem.

5.25. a) As is clear from the solution of Problem 5.24 a), it is possible to select points A, B and C on edges a, b and c such that rays α, β and γ are not parallel to lines BC, CA and AB and intersect these lines at points A_1, B_1 and C_1, respectively. Denote for brevity the dihedral angles between lines ab and $a\alpha$, ac and $a\alpha$ by U and V, respectively; denote the angles between rays b and α, c and α by u and v, respectively; let us also denote the area of triangle XYZ by (XYZ).

Let us compute the volume of tetrahedron $SABA_1$ in two ways. On the one hand,

$$V_{SABA_1} = \frac{(SA_1B) \cdot h_a}{3} = \frac{SA_1 \cdot SB \cdot h_a \sin u}{6},$$

where h_a is the height dropped from vertex A to face SBC. On the other hand,

$$V_{SABA_1} = \frac{2}{3}\frac{(SAB)\cdot(SAA_1)\sin U}{SA} \quad \text{(cf. Problem 3.3)}.$$

Let

$$\frac{SA_1\cdot SB\cdot h_a\sin u}{6} = \frac{2(SAB)\cdot(SAA_1)\sin U}{3SA}.$$

Similarly,

$$\frac{SA_1\cdot SC\cdot h_a\sin v}{6} = \frac{2(SAC)\cdot(SAA_1)\sin V}{3SA}.$$

By dividing one of these equalities by another one, we get

$$\frac{SB}{SC}\cdot\frac{\sin u}{\sin v} = \frac{(SAB)}{(SAC)}\cdot\frac{\sin U}{\sin V}.$$

This equality means that

$$\frac{SB}{SC}\cdot\frac{\sin(b,\alpha)}{\sin(c,\alpha)} = \frac{(SAB)}{(SAC)}\cdot\frac{\sin(ab,a\alpha)}{\sin(ac,a\alpha)}$$

(one only has to verify that the signs of these expressions coincide). By applying similar arguments to points B_1 and C_1 and multiplying the obtained identities we get the required identity after a simplification.

b) To solve this problem, we have to make use of the results of Problems 5.24 a) and 5.25 a).

c) To solve this problem one has to make use of the results of Problems 5.24 b) and 5.25 a).

5.26. Let a, b and c be edges SA, SB and SC, respectively; α, β and γ rays SA_1, SB_1 and SC_1, respectively. Since $\angle ASB_1 = \angle ASC_1$, it follows that $|\sin(a,\beta)| = |\sin(a,\gamma)|$. Similarly, $|\sin(b,\alpha)| = |\sin(b,\gamma)|$ and $|\sin(c,\alpha)| = |\sin(c,\beta)|$. Hence,

$$\left|\frac{\sin(a,\gamma)}{\sin(b,\gamma)}\cdot\frac{\sin(b,\alpha)}{\sin(c,\alpha)}\cdot\frac{\sin(c,\beta)}{\sin(a,\beta)}\right| = 1.$$

It is also clear that each of the three factors here is negative; hence, their product is equal to -1. It remains to make use of the first Ceva's theorem (Problem 5.24 b)).

5.27. It is easy to verify that

$$\sin(a,\gamma) = -\sin(b,\gamma'),\quad \sin(b,\gamma) = -\sin(a,\gamma'),\quad \sin(b,\alpha) = -\sin(c,\alpha'),$$
$$\sin(c,\alpha) = -\sin(b,\alpha'),\quad \sin(c,\beta) = -\sin(a,\beta'),\quad \sin(a,\beta) = -\sin(c,\beta').$$

Therefore,

$$\frac{\sin(a,\gamma')}{\sin(b,\gamma')}\cdot\frac{\sin(b,\alpha')}{\sin(c,\alpha')}\cdot\frac{\sin(c,\beta')}{\sin(a,\beta')} = \left(\frac{\sin(a,\gamma)}{\sin(b,\gamma)}\cdot\frac{\sin(b,\alpha)}{\sin(c,\alpha)}\cdot\frac{\sin(c,\beta)}{\sin(a,\beta)}\right)^{-1}.$$

To solve headings a) and b) it suffices to make use of this identity and the first theorems of Menelaus and Ceva (Problems 5.24 a) and 5.24 b)).

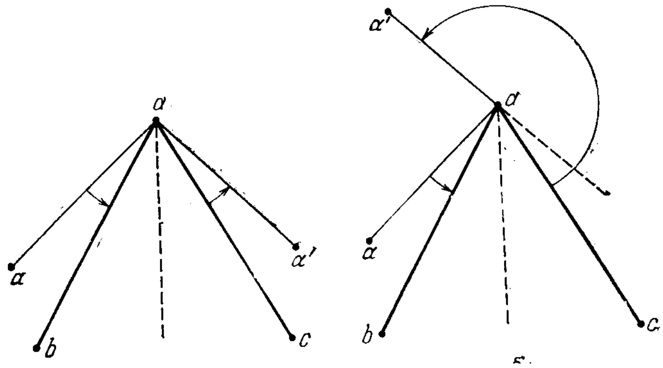

FIGURE 43 (SOL. 5.28)

5.28. Let us consider the section by the plane passing through edge a perpendicularly to it and let us denote the intersection points of the given lines and edges with this plane by the same letters as the lines and edges themselves. The two cases are possible:

1) Rays $a\alpha$ and $a\alpha'$ are symmetric through the bisector of angle bac (Fig. 43 a)).

2) Rays $a\alpha$ and $a\alpha'$ are symmetric through a line perpendicular to the bisector of the angle bac (Fig. 43 b)).

In the first case the angle of rotation from ray $a\alpha$ to ray ab is equal to the angle of rotation from ray ac to ray $a\alpha'$ and the angle of rotation from ray $a\alpha$ to ray ac is equal to the ray of rotation from ray ab to ray $a\alpha$.

In the second case these angles are not equal but differ by 180°. Passing to the angles between halfplanes we get:

in the first case, $\sin(ab, a\alpha) = -\sin(ac, a\alpha')$ and $\sin(ac, a\alpha) = -\sin(ab, a\alpha')$;

in the second case, $\sin(ab, a\alpha) = \sin(ac, a\alpha')$ and $\sin(ac, a\alpha) = \sin(ab, a\alpha')$.

In both cases

$$\frac{\sin(ab, a\alpha)}{\sin(ac, a\alpha)} = \frac{\sin(ac, a\alpha')}{\sin(ab, a\alpha')}.$$

By performing similar arguments for the edges b and c and by multiplying all these identities we get

$$\frac{\sin(ab, a\alpha)}{\sin(ac, a\alpha)} \cdot \frac{\sin(bc, b\beta)}{\sin(ba, b\beta)} \cdot \frac{\sin(ca, c\gamma)}{\sin(cb, c\gamma)} = \left(\frac{\sin(ab, a\alpha')}{\sin(ac, a\alpha')} \cdot \frac{\sin(bc, b\beta')}{\sin(ba, b\beta')} \cdot \frac{\sin(ca, c\gamma')}{\sin(cb, c\gamma')}\right)^{-1}.$$

To solve headings a) and b) it suffices to make use of this identity and second theorems of Menelaus and Ceva (problems 5.25 b) and 5.25 c)).

5.29. Denote by π_{ij} the plane symmetric to plane PA_iA_j through the bisector plane of the dihedral angle at edge A_iA_j. As follows from Problem 5.28 b), plane π_{il} passes through the intersection line of planes π_{ij} and π_{ik}. Let us consider three planes: π_{12}, π_{23} and π_{31}. Two cases are possible:

1) These planes have a common point P^*. Then planes π_{14}, π_{24} and π_{34} pass through lines A_1P^*, A_2P^* and A_3P^*, respectively, i.e., all the 6 planes π_{ij} pass through point P^*.

2) Planes π_{12} and π_{13}, π_{12} and π_{23}, π_{31} and π_{32} intersect along lines l_1, l_2, l_3, respectively, and lines l_1, l_2, l_3 are parallel to each other. Then planes π_{14}, π_{24} and π_{34} pass through lines l_1, l_2 and l_3, respectively, i.e., all the six planes π_{ij} are parallel to one line.

5.30. The projection to plane BSC of any line l passing through point S coincides with the line along which the plane that passes through edge SA and line l intersects plane BSC. Therefore, it suffices to prove that planes drawn through edge SA and the intersection lines of planes π_b and π_c, π'_b and π'_c are symmetric through the bisector plane of the dihedral angle at edge SA. This follows from the result of Problem 5.25 c).

5.31. a) In the solution of this problem we will make use of the fact that *the projection D_1 of point D to plane ABC lies on the circle circumscribed about triangle ABC* (Problem 7.32 b)).

In triangles DAB, DBC and DAC draw heights DC_1, DA_1 and DB_1. We have to show that rays DA_1, DB_1 and DC_1 lie in one plane, i.e., points A_1, B_1 and C_1 lie on one line. Since line DD_1 is perpendicular to plane ABC, it follows that $DD_1 \perp A_1C$. Moreover, $DA_1 \perp A_1C$. Therefore, line A_1C is perpendicular to plane DD_1A_1; in particular, $D_1A_1 \perp A_1C$. Therefore, A_1, B_1 and C_1 are the bases of the perpendiculars dropped to lines BC, CA and AB, respectively, from point D_1 that lies on the circle circumscribed about triangle ABC.

(For points B_1 and C_1 the proof is carried out in the same way as for point A_1.)

It is possible to prove that points A_1, B_1 and C_1 lie on one line (see Problem 2.29).

b) If AA_1 is the height of triangle ABC and O the center of its circumscribed circle, then rays AA_1 and AO are symmetric through the bisector of angle BAC. Indeed, it is easy to verify that

$$\angle BAO = \angle CAA_1 = |90^\circ - \angle C|$$

(one has to consider two cases: when angle C is an obtuse one and when it is an acute one). Since, as has been proved in the preceding heading, the lines that connect vertex D with the intersection points of the heights of faces DAB, DBC and DAC lie in one plane, it follows that the lines that connect vertex D with the centers of circumscribed circles of faces DAB, DBC and DAC also lie in one plane (cf. Problem 5.27 a)).

CHAPTER 6. TETRAHEDRON, PYRAMID, PRISM

§1. Properties of tetrahedrons

6.1. Is it true for any tetrahedron that its heights meet at one point?

6.2. a) Through vertex A of tetrahedron $ABCD$ there are drawn 3 planes perpendicular to the opposite edges. Prove that these planes intersect along one line.

b) Through each vertex of tetrahedron the plane perpendicular to the opposite face and containing the center of its circumscribed circle is drawn. Prove that these four planes intersect at one point.

6.3. A median of the tetrahedron is a segment that connects a vertex of the tetrahedron with the intersection point of the medians of the opposite face. Express the length of the median of the tetrahedron in terms of the lengths of the tetrahedron's edges.

6.4. Prove that the center of the sphere inscribed in a tetrahedron lies inside the tetrahedron formed by the tangent points.

6.5. Consider a tetrahedron. Let S_1 and S_2 be the areas of the tetrahedron's faces adjacent to edge a; let α be the dihedral angle at this edge; b the edge opposite to a; let φ be the angle between b and a. Prove that

$$S_1^2 + S_2^2 - 2S_1S_2\cos\alpha = \frac{1}{4}(ab\sin\varphi)^2.$$

6.6. Prove that the product of the lengths of two opposite edges of the tetrahedron divided by the product of sines of the dihedral angles at these edges is the same for all the three pairs of the opposite edges of the tetrahedron. (*The law of sines for a tetrahedron.*)

6.7. a) Let S_1, S_2, S_3 and S_4 be the areas of the faces of a tetrahedron; P_1, P_2 and P_3 the areas of the faces of the parallelepiped whose faces pass through the edges of the tetrahedron parallel to its opposite edges. Prove that

$$S_1^2 + S_2^2 + S_3^2 + S_4^2 = P_1^2 + P_2^2 + P_3^2.$$

b) Let h_1, h_2, h_3 and h_4 be the heights of the tetrahedron, d_1, d_2 and d_3 the distances between its opposite edges. Prove that

$$\frac{1}{h_1^2} + \frac{1}{h_2^2} + \frac{1}{h_3^2} + \frac{1}{h_4^2} = \frac{1}{d_1^2} + \frac{1}{d_2^2} + \frac{1}{d_3^2}.$$

6.8. Let S_i, R_i and l_i ($i = 1, 2, 3, 4$) be the areas of the faces, the radii of the disks circumscribed about these faces and the distances from the centers of these disks to the opposite vertices of the tetrahedron, respectively. Prove that

$$18V^2 = \sum_{i=1}^{4} S_i^2(l_i^2 - R_i^2),$$

where V is the volume of the tetrahedron.

6.9. Prove that for any tetrahedron there exists a triangle the lengths of whose sides are equal to the products of the lengths of the opposite edges of the tetrahedron and the area S of this triangle is equal to $6VR$, where V is the volume of the tetrahedron, R is the radius of its circumscribed sphere. (*Krell's formula*).

6.10. Let a and b be the lengths of two skew edges of a tetrahedron, α and β the dihedral angles at these edges. Prove that the quantity

$$a^2 + b^2 + 2ab \cot\alpha \cot\beta$$

does not depend on the choice of the pair of skew edges. (*Bretshneider's theorem*).

6.11. Prove that for any tetrahedron there exists not less than 5 and not more than 8 spheres each of which is tangent to all the planes of its faces.

§2. Tetrahedrons with special properties

6.12. In triangular pyramid $SABC$ with vertex S the lateral edges are equal and the sum of dihedral angles at the edges SA and SC is equal to 180°. Express the length of the lateral edge through the sides a and c of triangle ABC.

6.13. The sum of the lengths of one pair of skew edges of a tetrahedron is equal to the sum of the lengths of another pair. Prove that the sum of dihedral angles at the first pair of edges is equal to the sum of dihedral angles at the second pair.

6.14. All the faces of a tetrahedron are right triangles similar to each other. Find the ratio of the longest edge to the shortest one.

6.15. The edge of a regular tetrahedron $ABCD$ is equal to a. The vertices of a spatial quadrilateral $A_1B_1C_1D_1$ lie on the corresponding faces of the tetrahedron (A_1 lies on the face opposite to A, etc.) and its sides are perpendicular to the faces of the tetrahedron: $A_1B_1 \perp BCD$, $B_1C_1 \perp CDA$, $C_1D_1 \perp DAB$ and $D_1A_1 \perp ABC$. Calculate the lengths of the sides of quadrilateral $A_1B_1CD_1$.

6.16. A sphere is tangent to edges AB, BC, CD and DA of tetrahedron $ABCD$ at points L, M, N and K, respectively; the tangent points are the vertices of a square. Prove that if the sphere is tangent to edge AC, then it is tangent to edge BD.

6.17. Let M be the center of mass of tetrahedron $ABCD$, O the center of its circumscribed sphere.

a) Prove that lines DM and OM are perpendicular if and only if

$$AB^2 + BC^2 + CA^2 = AD^2 + BD^2 + CD^2.$$

b) Prove that if points D and M and the intersection points of the medians of the faces at vertex D lie on one sphere, then $DM \perp OM$.

§3. A rectangular tetrahedron

6.18. In tetrahedron $ABCD$, the plane angles at vertex D are right ones. Let $\angle CAD = \alpha$, $\angle CBD = \beta$ and $\angle ACB = \varphi$. Prove that $\cos\varphi = \sin\alpha \sin\beta$.

6.19. All the plane angles at one vertex of a tetrahedron are right ones. Prove that the lengths of segments that connect the midpoints of the opposite edges are equal.

6.20. In tetrahedron $ABCD$, the plane angles at vertex D are right ones. Let h be the height of the tetrahedron dropped from vertex D; let a, b and c be the lengths of the edges going from vertex D. Prove that

$$\frac{1}{h^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

6.21. In tetrahedron $ABCD$ the plane angles at vertex A are right ones and $AB = AC + AD$. Prove that the sum of plane angles at vertex B is equal to 90°.

6.22. Three dihedral angles of a tetrahedron are right ones. Prove that this tetrahedron has three plane right angles.

6.23. In a tetrahedron, three dihedral angles are right ones. One of the segments that connects the midpoints of the opposite edges is equal to a, another one to b and $b > a$. Find the length of the longest edge of the tetrahedron.

6.24. Three dihedral angles of a tetrahedron not belonging to one vertex are equal to 90° and the remaining dihedral angles are equal to each other. Find these angles.

§4. Equifaced tetrahedrons

A tetrahedron is called an *equifaced* one if all its faces are equal, i.e., its opposite edges are pairwise equal.

6.25. Prove that all the faces of a tetrahedron are equal if and only if one of the following conditions holds:

a) the sum of the plane angles at a vertex is equal to 180° and, moreover, there are two pairs of equal opposite edges;

b) the centers of the inscribed and circumscribed spheres coincide;

c) the radii of the circles circumscribed about the faces are equal;

d) the center of mass and the center of the circumscribed sphere coincides.

6.26. In tetrahedron $ABCD$, the dihedral angles at edges AB and DC are equal; the dihedral angles at edges BC and AD are also equal. Prove that $AB = DC$ and $BC = AD$.

6.27. The line that passes through the center of mass of tetrahedron $ABCD$ and the center of its circumscribed sphere intersects edges AB and CD. Prove that $AC = BD$ and $AD = BC$.

6.28. The line that passes through the center of mass of tetrahedron $ABCD$ and the center of one of its escribed spheres intersects edges AB and CD. Prove that $AC = BD$ and $AD = BC$.

6.29. Prove that if

$$\angle BAC = \angle ABD = \angle ACD = \angle BDC$$

then tetrahedron $ABCD$ is an equifaced one.

6.30. Given tetrahedron $ABCD$; let O_a, O_b, O_c and O_d be the centers of the escribed spheres tangent to its faces BCD, ACD, ABD and ABC, respectively. Prove that if trihedral angles O_aBCD, O_bACD, O_cABD and O_dABC are right ones, then all the faces of the given tetrahedron are equal.

Remark. There are also other conditions that distinguish equifaced tetrahedrons; see, for example, Problems 2.32, 6.48 and 14.22.

6.31. Edges of an equifaced tetrahedron are equal to a, b and c. Compute its volume V and the radius R of the circumscribed sphere.

6.32. Prove that for an equifaced tetrahedron

a) the radius of the inscribed ball is a half of the radius of the ball tangent to one of the faces of tetrahedron and extensions of the three other faces;

b) the centers of the four escribed balls are the vertices of the tetrahedron equal to the initial one.

6.33. In an equifaced tetrahedron $ABCD$ height AH is dropped; H_1 is the intersection point of the heights of face BCD; h_1 and h_2 are the lengths of the segments into which point H_1 divides one of the heights of face BCD.

a) Prove that points H and H_1 are symmetric through the center of the circumscribed circle of triangle BCD.

b) Prove that $AH^2 = 4h_1h_2$.

6.34. Prove that in an equifaced tetrahedron the bases of the heights, the midpoints of the heights and the intersection points of the faces' heights all belong to one sphere (*the sphere of* 12 *points*).

6.35. a) Prove that the sum of the cosines of dihedral angles of an equifaced tetrahedron is equal to 2.

b) The sum of the plane angles of a trihedral angle is equal to 180°. Find the sum of the cosines of its dihedral angles.

§5. Orthocentric tetrahedrons

A tetrahedron is called an *orthocentric* one if all its heights (or their extensions) meet at one point.

6.36. a) Prove that if $AD \perp BC$, then the heights dropped from vertices B and C (as well as the heights dropped from vertices A and D) intersect at one point and this point lies on the common perpendicular to AD and BC.

b) Prove that if the heights dropped from vertices B and C intersect at one point, then $AD \perp BC$ (consequently, the heights dropped from vertices A and D also intersect at one point).

c) Prove that a tetrahedron is an orthocentric one if and only if two pairs of its opposite edges are perpendicular to each other (in this case the third pair of its opposite edges is also perpendicular to each other).

6.37. Prove that in an orthocentric tetrahedron the common perpendiculars to the pairs of opposite edges intersect at one point.

6.38. Let K, L, M and N be the midpoints of edges AB, BC, CD and DA of tetrahedron $ABCD$.

a) Prove that $AC \perp BD$ if and only if $KM = LN$.

b) Prove that the tetrahedron is an orthocentric one if and only if the segments that connect the midpoints of opposite edges are equal.

6.39. a) Prove that if $BC \perp AD$, then the heights dropped from vertices A and D to line BC have the same base.

b) Prove that if the heights dropped from vertices A and D to line BC have the same base, then $BC \perp AD$ (hence, the heights dropped from vertices B and C to line AD also have the same base).

6.40. Prove that a tetrahedron is an orthocentric one if and only if one of the following conditions holds:

a) the sum of squared lengths of the opposite edges are equal;

b) the products of the cosines of the opposite dihedral angles are equal;

c) the angles between the opposite edges are equal.

Remark. There are also other conditions that single out orthocentric tetrahedrons: see, for example, Problems 2.11 and 7.1.

6.41. Prove that in an orthocentric tetrahedron:

a) all the plane angles at one vertex are simultaneously either acute, or right, or obtuse;

b) one of the faces is an acute triangle.

6.42. Prove that in an orthocentric triangle the relation

$$OH^2 = 4R^2 - 3d^2$$

holds, where O is the center of the circumscribed sphere, H the intersection point of the heights, R the radius of the circumscribed sphere, d the distance between the midpoints of the opposite edges.

6.43. a) Prove that the circles of 9 points of triangles ABC and DBC belong to one sphere if and only if $BC \perp AD$.

b) Prove that for an orthocentric triangle circles of 9 points of all its faces belong to one sphere (*the sphere of* 24 *points*).

c) Prove that if $AD \perp BC$, then the sphere that contains circles of 9 points of triangles ABC and DBC and the sphere that contains circles of 9 points of triangles ABD and CBD intersect along a circle that lies in the plane that divides the common perpendicular to BC and AD in halves and is perpendicular to it.

6.44. Prove that in an orthocentric tetrahedron the centers of mass of faces, the intersection points of the heights of faces, and the points that divide the segments that connect the intersection point of the heights with the vertices in ratio $2:1$ counting from the vertex lie on one sphere (*the sphere of* 12 *points*).

6.45. a) Let H be the intersection point of heights of an orthocentric tetrahedron, M' the center of mass of a face, N the intersection point of ray HM' with the tetrahedron's circumscribed sphere. Prove that $HM' : M'N = 1 : 2$.

b) Let M be the center of mass of an orthocentric tetrahedron, H' the intersection point of heights of a face, N the intersection point of ray $H'M$ with the tetrahedron's circumscribed sphere. Prove that $H'M : MN = 1 : 3$.

6.46. Prove that in an orthocentric tetrahedron Monge's point (see Problem 7.32 a)) coincides with the intersection point of heights.

§6. Complementing a tetrahedron

By drawing a plane through every edge of a tetrahedron parallel to the opposite edge we can complement the tetrahedron to a parallelepiped (Fig. 44).

6.47. Three segments not in one plane intersect at point O that divides each of them in halves. Prove that there exist exactly two tetrahedrons in which these segments connect the midpoints of the opposite edges.

6.48. Prove that all the edges of a tetrahedron are equal if and only if one of the following conditions holds:

a) by complementing the tetrahedron we get a rectangular parallelepiped;

b) the segments that connect the midpoints of the opposite edges are perpendicular to each other;

c) the areas of all the faces are equal;

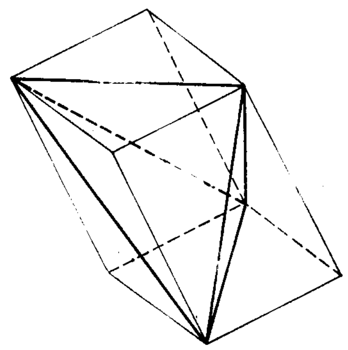

FIGURE 44 (§6)

d) the center of mass and the center of an escribed sphere coincide.

6.49. Prove that in an equifaced tetrahedron all the plane angles are acute ones.

6.50. Prove that the sum of squared lengths of the edges of a tetrahedron is equal to four times the sum of the squared distances between the midpoints of its opposite edges.

6.51. Let a and a_1, b and b_1, c and c_1 be the lengths of the opposite edges of a tetrahedron; α, β, γ the corresponding angles between them $(\alpha, \beta, \gamma \leq 90^\circ)$. Prove that one of the three numbers $aa_1 \cos\alpha$, $bb_1 \cos\beta$ and $cc_1 \cos\gamma$ is the sum of the other two ones.

6.52. Line l passes through the midpoints of edges AB and CD of tetrahedron $ABCD$; a plane Π that contains l intersects edges BC and AD at points M and N. Prove that line l divides segment MN in halves.

6.53. Prove that lines that connect the midpoint of a height of a regular tetrahedron with vertices of the face onto which this height is dropped are pairwise perpendicular.

§7. Pyramid and prism

6.54. The planes of lateral faces of a triangular pyramid constitute equal angles with the plane of the base. Prove that the projection of the height to the plane of the base is the center of the inscribed or escribed circle at the base.

6.55. In a triangular pyramid the trihedral angles at edges of the base are equal to α. Find the volume of the pyramid if the lengths of the edges at the base are equal to a, b and c.

6.56. On the base of a triangular pyramid $SABC$, a point M is taken and lines parallel to edges SA, SB and SC and intersecting lateral faces at points A_1, B_1 and C_1 are drawn through M. Prove that

$$\frac{MA_1}{SA} + \frac{MB_1}{SB} + \frac{MC_1}{SC} = 1.$$

6.57. Vertex S of triangular pyramid $SABC$ coincides with the vertex of a circular cone and points A, B and C lie on the circle of its base. The dihedral angles at edges SA, SB and SC are equal to α, β and γ. Find the angle between plane SBC and the plane tangent to the surface of the cone along the generator SC.

6.58. Similarly directed vectors $\{AA_1\}$, $\{BB_1\}$ and $\{CC_1\}$ are perpendicular to plane ABC and their lengths are equal to the corresponding heights of triangle ABC the radius of whose inscribed circle is equal to r.

a) Prove that the distance from the intersection point M of planes A_1BC, AB_1C and ABC_1 to plane ABC is equal to r.

b) Prove that the distance from the intersection point N of planes A_1B_1C, A_1BC_1 and AB_1C_1 to plane ABC is equal to $2r$.

* * *

6.59. In a regular truncated quadrangular pyramid with height of the lateral face equal to a a ball can be inscribed. Find the area of the pyramid's lateral surface.

6.60. The perpendicular to the base of a regular pyramid at point M intersects the planes of lateral faces at points $M_1, \dots, M_n$. Prove that the sum of the lengths of segments $MM_1, \dots, MM_n$ is the same for all points M from the base of the pyramid.

6.61. A ball is inscribed into an n-gonal pyramid. The lateral faces of the pyramid are rotated about the edges of the base and arranged in the plane of the base so that they lie on the same side with respect to the corresponding edges together with the base itself. Prove that the vertices of these faces distinct from the vertices of the base lie on one circle.

6.62. From the vertices of the base of the inscribed pyramid the heights are drawn in the lateral faces. Prove that the lines that connect the basis of the heights in each face are parallel to one plane. (The plane angles at the vertex of the pyramid are supposed to be not right ones.)

6.63. The base of a pyramid with vertex S is a parallelogram $ABCD$. Prove that the lateral edges of the pyramid form equal angles with ray SO that lies inside the tetrahedral angle $SABCD$ if and only if

$$SA + SC = SB + SD.$$

6.64. The bases of a truncated quadrangular pyramid $ABCDA_1B_1C_1D_1$ are parallelograms $ABCD$ and $A_1B_1C_1D_1$. Prove that any line that intersects three of the four lines AB, BC_1, CD_1 and DA_1 either intersects the fourth line or is parallel to it.

* * *

6.65. Find the area of the total surface of the prism circumscribed about a sphere if the area of the base of the prism is equal to S.

6.66. On the lateral edges BB_1 and CC_1 of a regular prism $ABCA_1B_1C_1$, points P and P_1 are taken so that

$$BP : PB_1 = C_1P : PC = 1 : 2.$$

a) Prove that the dihedral angles at edges AP_1 and A_1P of tetrahedron AA_1PP_1 are right ones.

b) Prove that the sum of dihedral angles at edges AP, PP_1 and P_1A_1 of tetrahedron AA_1PP_1 is equal to $180°$.

Problems for independent study

6.67. In a prism (not necessarily right one) a ball is inscribed.

a) Prove that the height of the prism is equal to the diameter of the ball.

b) Prove that the tangent points of the ball with the lateral faces lie in one plane and this plane is perpendicular to the lateral edges of the prism.

6.68. A sphere is tangent to the lateral faces of a prism at the centers of the circles circumscribed about them; the plane angles at the vertex of this prism are equal. Prove that the prism is a regular one.

6.69. A sphere is tangent to the three sides of the base of a triangular pyramid at their midpoints and intersects the lateral edges at their midpoints. Prove that the pyramid is a regular one.

6.70. The sum of the lengths of the opposite edges of tetrahedron $ABCD$ is the same for any pair of opposite edges. Prove that the inscribed circles of any two faces of the tetrahedron are tangent to the common edge of these faces at one point.

6.71. Prove that if the dihedral angles of a tetrahedron are equal, then this tetrahedron is a regular one.

6.72. In a triangular pyramid $SABC$, angle $\angle BSC$ is a right one and $\angle ASC = \angle ASB = 60^\circ$. Vertices A and S and the midpoints of edges SB, SC, AB and AC lie on one sphere. Prove that edge SA is a diameter of the sphere.

6.73. In a regular hexagonal pyramid, the center of the circumscribed sphere lies on the surface of the inscribed sphere. Find the ratio of radii of the inscribed and circumscribed spheres.

6.74. In a regular quadrangular pyramid, the center of the circumscribed sphere lies on the surface of the inscribed one. Find the value of the plane angle at the vertex of the pyramid.

6.75. The base of triangular prism $ABCA_1B_1C_1$ is an isosceles triangle. It is known that pyramids $ABCC_1$, ABB_1C_1 and $AA_1B_1C_1$ are equal. Find the dihedral angles at the edges of the base of the prism.

Solutions

6.1. No, not for any tetrahedron. Consider triangle ABC in which angle $\angle A$ is not a right one and erect perpendicular AD to the plane of the triangle. In tetrahedron $ABCD$, the heights drawn from vertices C and D do not intersect.

6.2. a) The perpendicular dropped from vertex A to plane BCD belongs to all the three given planes.

b) It is easy to verify that all the indicated planes pass through the center of the circumscribed sphere of the tetrahedron.

6.3. Let $AD = a$, $BD = b$, $CD = c$, $BC = a_1$, $CA = b_1$ and $AB = c_1$. Compute the length m of median DM. Let N be the midpoint of edge BC, $DN = p$ and $AN = q$. Then

$$DM^2 + MN^2 - 2DM \cdot MN \cos DMN = DN^2$$

and

$$DM^2 + AM^2 - 2DM \cdot AM \cos DMA = AD^2$$

and, therefore,

$$(*) \qquad m^2 + \frac{q^2}{9} - \frac{2mq\cos\varphi}{3} = p^2 \text{ and } m^2 + \frac{4q^2}{9} + \frac{4mq\cos\varphi}{3} = a^2.$$

By multiplying the first of equalities $(*)$ by 2 and adding it to the second equality in $(*)$ we get

$$3m^2 = a^2 + 2p^2 - \frac{2q^2}{3}.$$

Since

$$p^2 = \frac{2b^2 + 2c^2 - a_1^2}{4} \text{ and } q^2 = \frac{2b_1^2 + 2c_1^2 - a_1^2}{4},$$

it follows that

$$9m^2 = 3(a^2 + b^2 + c^2) - a_1^2 - b_1^2 - c_1^2.$$

6.4. It suffices to prove that if the sphere is inscribed in the trihedral angle, then the plane passing through the tangent points separates vertex S of the trihedral angle from the center O of the inscribed sphere. The plane that passes through the tangent points coincides with the plane that passes through the circle along which the cone with vertex S is tangent to the given sphere. Clearly, this plane separates points S and O; to prove this, we can consider any section that passes through S and O.

6.5. The projection of the tetrahedron to the plane perpendicular to edge a is a triangle with sides $\frac{2S_1}{a}$, $\frac{2S_2}{a}$ and $b \sin\varphi$; the angle between the first two sides is equal to α. Expressing the law of cosines for this triangle we get the required statement.

6.6. Consider tetrahedron $ABCD$. Let $AB = a$, $CD = b$; let α and β be the dihedral angles at edges AB and CD; S_1 and S_2 be the areas of faces ABC and ABD, S_3 and S_4 the areas of faces CDA and CDB; V the volume of the tetrahedron. By Problem 3.3

$$V = \frac{2S_1S_2 \sin\alpha}{3a} \text{ and } V = \frac{2S_3S_4 \sin\beta}{3b}.$$

Hence,

$$\frac{ab}{\sin\alpha \sin\beta} = \frac{4S_1S_2S_3S_4}{9V^2}.$$

6.7. a) Let α, β and γ be the dihedral angles at the edges of the face with area S_1. Then

$$S_1 = S_2 \cos\alpha + S_3 \cos\beta + S_4 \cos\gamma$$

(cf. Problem 2.13). Moreover, thanks to Problem 6.5

$$\begin{gathered} S_1^2 + S_2^2 - 2S_1S_2 \cos\alpha = P_1^2, \\ S_1^2 + S_3^2 - 2S_1S_3 \cos\beta = P_2^2, \\ S_1^2 + S_4^2 - 2S_1S_4 \cos\gamma = P_3^2. \end{gathered}$$

Therefore,

$$\begin{aligned} P_1^2 + P_2^2 + P_3^2 = S_2^2 + S_3^2 + S_4^2 + 3S_1^2 - 2S_1(S_2 \cos\alpha + S_3 \cos\beta + S_4 \cos\gamma) = \\ = S_1^2 + S_2^2 + S_3^2 + S_4^2. \end{aligned}$$

b) By dividing both parts of the equality obtained in heading a) by $9V^2$, where V is the volume of the tetrahedron, we get the desired statement.

6.8. First, let us carry out the proof for the case when the center of the circumscribed ball lies inside the tetrahedron. First of all, let us prove that

$$l_i^2 - R_i^2 = 2h_i d_i,$$

where d_i is the distance from the center of the circumscribed ball to the i-th face, h_i the height of the tetrahedron dropped to this face. For definiteness sake we will assume that the index i corresponds to face ABC.

Let O be the center of the circumscribed sphere of tetrahedron $ABCD$, O_1 the projection of O to face ABC, DH the height of , H_1 the projection of O to DH. Then

$$O_1H^2 = DO_1^2 - DH^2 = l_i^2 - h_i^2;$$
$$OH_1^2 = DO^2 - DH_1^2 = R^2 - (h_i - d_i)^2 = R^2 - d_i^2 + 2h_i d_i - h_i^2,$$

where R is the radius of the circumscribed sphere of the tetrahedron. Since $O_1H = OH_1$, it follows that $l_i^2 - R^2 + d_i^2 = 2h_i d_i$. It remains to notice that

$$R_i^2 = AO_1^2 = AO^2 - OO_1^2 = R^2 - d_i^2.$$

The following transformations complete the proof:

$$\sum S_i^2(l_i^2 - R_i^2) = \sum 2S_i^2 h_i d_i = \sum 2S_i^2 h_i^2 \frac{d_i}{h_i} = 18V^2 \sum \frac{d_i}{h_i}.$$

By Problem 8.1.b) $\sum \frac{d_i}{h_i} = 1$.

If the center of the circumscribed ball lies outside the tetrahedron our arguments practically do not change: one only has to assume one of the quantities d_i to be negative.

6.9. Let the lengths of edges AD, BD and CD be equal to a, b and c, respectively; let the lengths of edges BC, CA and AB be equal to a', b' and c', respectively. Through vertex D, let us draw a plane Π tangent to the sphere circumscribed about the tetrahedron. Consider tetrahedron A_1BC_1D formed by planes Π, BCD, ABD and the plane that passes through the vertex B parallel to plane ACD and tetrahedron AB_2C_2D formed by planes Π, ABD, ACD and the plane that passes through vertex A parallel to plane BCD (Fig. 45).

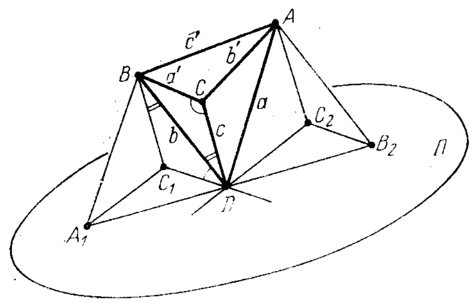

FIGURE 45 (SOL. 6.9)

Since DC_1 is the tangent to the circle circumscribed about triangle DBC, it follows that $\angle BDC_1 = \angle BCD$. Moreover, $BC_1 \parallel CD$, therefore, $\angle C_1BD = \angle BDC$. Hence, $\triangle DC_1B \sim \triangle CBD$ and, therefore, $DC_1 : DB = CB : CD$, i.e., $DC_1 = \frac{a'b}{c}$. Similarly, $DA_1 = \frac{c'b}{a}, DC_2 = \frac{b'a}{c}$ and $DB_2 = \frac{c'a}{b}$. Since $\triangle A_1C_1D \sim \triangle DC_2B_2$, it follows that $A_1C_1 : A_1D = DC_2 : DB_2$, i.e., $A_1C_1 = \frac{b'b^2}{ac}$.

Thus, the lengths of the sides of triangle A_1C_1D multiplied by $\frac{ac}{b}$ are equal to $a'a$, $b'b$ and $c'c$, respectively, and, therefore,

$$S_{A_1C_1D} = \frac{b^2}{a^2c^2}S.$$

Now, let us find the volume of tetrahedron A_1BC_1D. To this end, let us consider diameter DM of the circumscribed sphere of the initial tetrahedron and the perpendicular BK dropped to plane A_1C_1D. It is clear that $BK \perp DK$ and $DM \perp DK$. From the midpoint O of segment DM drop perpendicular OL to segment DB. Since $\triangle BDK \sim \triangle DOL$, it follows that $BK : BD = DL : DO$, i.e., $BK = \frac{b^2}{2R}$. Hence,

$$V_{A_1BC_1D} = \frac{1}{3}BK \cdot S_{A_1C_1D} = \frac{b^4}{6Ra^2c^2}S.$$

The ratio of volumes of tetrahedrons A_1BC_1D and $ABCD$ is equal to the product of ratios of the areas of faces BC_1D and BCD divided by the ratio of the lengths of the heights dropped to these faces; the latter ratio is equal to $S_{A_1BD} : S_{ABD}$. Since $\triangle DB_1B \sim \triangle CBD$, we have:

$$S_{BC_1D} : S_{BCD} = (DB : CD)^2 = b^2 : c^2.$$

Similarly,

$$S_{A_1BD} : S_{ABD} = b^2 : a^2.$$

Therefore,

$$V = \frac{a^2c^2}{b^4}V_{A_1BC_1D} = \frac{a^2c^2}{b^4} \cdot \frac{b^4}{6Ra^2c^2}S = \frac{S}{6R}.$$

6.10. Let S_1 and S_2 be the areas of faces with common edge a, S_3 and S_4 the areas of faces with common edge b. Further, let a, m and n be the lengths of the edges of the face of area S_1; let α, γ and δ be the values of the dihedral angles at these edges, respectively; h_1 the length of the height dropped to this face; H the base of this height; V the volume of the tetrahedron.

By connecting point H with the vertices of face S_1 (we will denote the face by the same letter as the one we used to denote its area) we get three triangles.

By expressing the area of face S_1 in terms of the areas of these triangles we get:

$$ah_1 \cot\alpha + mh_1 \cot\gamma + nh_1 \cot\delta = 2S_1.$$

(Since angles α, γ and δ vary from 0° to 180°, this formula remains true even if H lies outside the face.) Taking into account that $h_1 = \frac{3V}{S_1}$ we get

$$a\cot\alpha + m\cot\gamma + n\cot\delta = \frac{2S_1^2}{3V}.$$

By adding up such equalities for faces S_1 and S_2 and subtracting from them the equalities for the other faces we get

$$a\cot\alpha - b\cot\beta = \frac{S_1^2+S_2^2-S_3^2-S_4^2}{3V}.$$

Let us square this equality, replace $\cot^2\alpha$ and $\cot^2\beta$ with $\frac{1}{\sin^2\alpha}-1$ and $\frac{1}{\sin^2\beta}-1$, respectively, and make use of the equalities

$$\frac{a^2}{\sin^2\alpha}=\frac{4S_1^2S_2^2}{9V^2},\quad \frac{b^2}{\sin^2\beta}=\frac{4S_3^2S_4^2}{9V^2},$$

(see Problem 3.3). We get

$$a^2+b^2+2ab\cot\alpha\cot\beta=\frac{2Q-T}{9V^2},$$

where Q is the sum of squared pairwise products of areas of the faces, T is the sum of the fourth powers of the areas of the faces.

6.11. Let V be the volume of the tetrahedron; S_1, S_2, S_3 and S_4 the areas of its faces. If the distance from point O to the i-th face is equal to h_i, then

$$\frac{\sum\varepsilon_i h_i S_i}{3}=V,$$

where $\varepsilon_i=+1$ if point O and the tetrahedron lie on one side of the i-th face and $\varepsilon_i=-1$ otherwise. Therefore, if r is the radius of the ball tangent to all the planes of the faces of the tetrahedron, then $\frac{(\sum\varepsilon_i S_i)r}{3}=V$, i.e., $\sum\varepsilon_i S_i>0$.

Conversely, if for a given collection of signs $\varepsilon_i=\pm1$ the value $\sum\varepsilon_i S_i$ is positive, then there exists a corresponding ball. Indeed, consider a point for which

$$h_1=h_2=h_3=r,\ \text{where } r=\tfrac{3V}{\sum\varepsilon_i S_i}$$

(in other words, we consider the intersection point of the three planes). For this point, h_4 is also equal to r.

For any tetrahedron there exists an inscribed ball ($\varepsilon_i=1$ for all i). Moreover, since (by Problem 10.22) the area of any face is smaller than the sum of the areas of the other faces, it follows that there exist 4 escribed balls each of which is tangent to one of the faces and the extensions of the other three faces (one of the numbers ε_i is equal to -1).

It is also clear that if $\sum\varepsilon_i S_i$ is positive for a collection $\varepsilon_i=\pm1$, then it is negative for the collection with opposite signs. Since there are $2^4=16$ collections altogether, there are not more than 8 balls. There will be precisely 8 of them if the sum of the areas of any two faces is not equal to the sum of areas of the other two faces.

6.12. On ray AS, take point A_1 so that $AA_1=2AS$. In pyramid SA_1BC the dihedral angles at edges SA_1 and SC are equal and $SA_1=SC$; hence, $A_1B=CB=a$. Triangle ABA_1 is a right one because its median BS is equal to a half of AA_1. Therefore,

$$AA_1^2=A_1B^2+AB^2=a^2+c^2,\ \text{ i.e., } AS=\tfrac{\sqrt{A^2+c^2}}{2}.$$

6.13. If the sum of edges AB and CD in tetrahedron $ABCD$ is equal to the sum of the lengths of edges BC and AD, then there exists a sphere tangent to these four edges in inner points (see Problem 8.30). Let O be the center of the sphere. Now, observe that if tangents XP and XQ are drawn from point X to the sphere centered at O, then points P and Q are symmetric through the plane that passes through line XO and the midpoint of segment PQ; hence, planes POX and QOX form equal angles with plane XPQ.

Let us draw four planes passing through point O and the considered edges of tetrahedron. They split each of the considered dihedral angles into 2 dihedral angles. We have shown above that the obtained dihedral angles adjacent to one face of the tetrahedron are equal. One of the obtained angles enters both of the considered sums of dihedral angles for each face of the tetrahedron.

6.14. Let a be the length of the longest edge of the tetrahedron. In both faces adjacent to this edge this edge is the hypothenuse. These faces are equal because similar rectangular triangles with a common hypothenuse are equal; let m and n be the lengths of the legs of these right triangles, b the length of the sixth edge of the tetrahedron. The following two cases are possible:

1) The edges of length m exit from the same endpoint of edge a, the edges of length n exit from the other endpoint. In triangle with sides m, m and b only the angle opposite to b can be a right one; moreover, in triangle with sides a, m and n the legs should also be equal, i.e., $m = n$. As a result we see that all the faces of the tetrahedron are equal.

2) From each endpoint of edge a one edge of length m and one edge of length n exits. Then if $a = b$ the tetrahedron is also an equifaced one.

Now, observe that an equifaced tetrahedron cannot have right plane angles (Problem 6.49). Therefore, only the second variant is actually possible and $b < a$. Let, for definiteness, $m \geq n$. Since triangles with sides a, m, n and m, n, b are similar and side n cannot be the shortest side of the second triangle, it follows that

$$a : m = m : n = n : b = \lambda > 1.$$

Taking this into account we get $a^2 = m^2 + n^2$; hence, $\lambda^4 = \lambda^2 + 1$, i.e., $\lambda = \sqrt{\frac{1+\sqrt{5}}{2}}$.

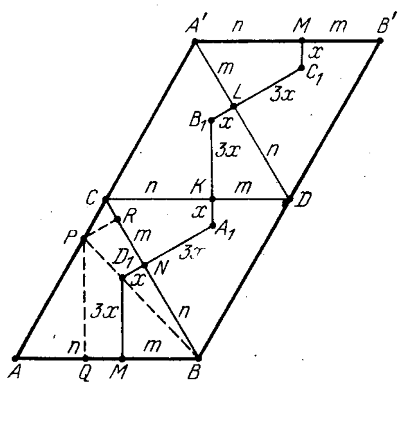

FIGURE 46 (SOL. 6.15)

6.15. Let us drop perpendiculars A_1K and B_1K to CD, B_1L and C_1L to AD, C_1M and D_1M to AB, D_1N and A_1N to BC. The ratios of the lengths of these perpendiculars are equal to the cosine of the dihedral angle at the edge of a regular tetrahedron, i.e., they are equal to $\frac{1}{3}$ (see Problem 2.14). Since the sides of quadrilateral $A_1B_1C_1D_1$ are perpendicular to the faces of a regular tetrahedron, their lengths are equal (see Problem 8.25). Hence,

$$A_1K = B_1L = C_1M = D_1N = x \quad \text{and} \quad B_1K = C_1L = D_1M = A_1N = 3x.$$

Let us consider the unfolding of the tetrahedron (Fig. 46). The edges of the tetrahedron are divided by points K, L, M and N into segments of length m and n. Since

$$x^2 + n^2 = D_1B^2 = 9x^2 + m^2,$$

it follows that

$$8x^2 = n^2 - m^2 = (n+m)(n-m) = a(n-m).$$

Let ray BD_1 intersect side AC at point P; let Q and R be the projections of point P to sides AB and BC, respectively. Since $PR : PQ = 1 : 3$, we have: $CP : PA = 1 : 3$. Therefore,

$$\frac{BR}{CB} = \frac{1}{2} + \frac{3}{8} = \frac{7}{8} \text{ and } \frac{BQ}{AB} = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}.$$

Hence,

$$\frac{n}{m} = \frac{BR}{BQ} = \frac{7}{5}$$

and, therefore, $x = \frac{a}{4\sqrt{3}}$. The lengths of the sides of quadrilateral $A_1B_1C_1D_1$ are equal to $2\sqrt{2}x = \frac{a}{\sqrt{6}}$.

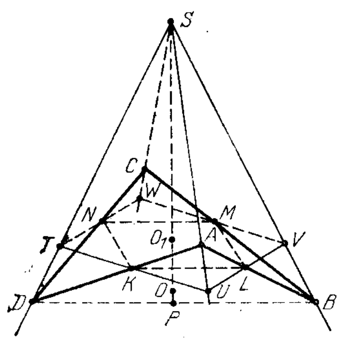

FIGURE 47 (SOL. 6.16)

6.16. By the hypothesis $KLMN$ is a square. Let us draw planes tangent to the sphere through points K, L, M and N. Since the angles of all these planes with plane $KLMN$ are equal, all these planes intersect at one point, S, lying on line OO_1, where O is the center of the sphere and O_1 is the center of the square.

These planes intersect the plane of the square $KLMN$ along the square $TUVW$ the midpoints of whose sides are points K, L, M and N (Fig. 47). In the tetrahedral angle $STUVW$ with vertex S all the plane angles are equal and points K, L, M and N lie on the bisectors of its plane angles, where

$$SK = SL = SM = SN.$$

Therefore, $SA = SC$ and $SD = SB$, hence, $AK = AL = CM = CN$ and $BL = BM = DN = DK$. By the hypothesis, AC is also tangent to the ball, hence,

$$AC = AK + CN = 2AK.$$

Since SK is the bisector of angle DSA, it follows that

$$DK : KA = DS : SA = DB : AC.$$

Now, the equality $AC = 2AK$ implies that $DB = 2DK$. Let P be the midpoint of segment DB; then P lies on line SO. Triangles DOK and DOP are equal because $DK = DP$ and $\angle DKO = 90^\circ = \angle DPO$. Therefore, $OP = OK = R$, where R is the radius of the sphere; it follows that DB is also tangent to the sphere.

6.17. a) Let $BC = a$, $CA = b$, $AB = c$, $DA = a_1$, $DB = b_1$ and $DC = c_1$. Further, let G be the intersection point of the medians of triangle ABC, N the intersections point of line DM with the circumscribed sphere, K the intersection point of line AG with the circle circumscribed about triangle ABC.

First, let us prove that

$$AG \cdot GK = \frac{a^2 + b^2 + c^2}{9}.$$

Indeed, $AG \cdot GK = R^2 - O_1G^2$, where R is the radius of the circumscribed circle of triangle ABC, where O_1 is its center. But

$$O_1G^2 = R^2 - \frac{a^2 + b^2 + c^2}{9}$$

(see \$). Further,

$$DG \cdot GN = AG \cdot GK = \frac{a^2 + b^2 + c^2}{9}$$

hence,

$$GN = \frac{a^2 + b^2 + c^2}{9m},$$

where

(1) $$m = DG = \frac{\sqrt{3(a_1^2 + b_1^2 + c_1^2) - a^2 - b^2 - c^2}}{3}$$

(see Problem 6.3). Therefore,

$$DN = DG + GN = m + \frac{a^2 + b^2 + c^2}{9m} = \frac{a_1^2 + b_1^2 + c_1^2}{3m}.$$

The fact that lines DM and OM are perpendicular is equivalent to the fact that $DN = 2DM$, i.e., $\frac{a_1^2+b_1^2+c_1^2}{3m} = \frac{3}{2}m$. Expressing m according to formula (1) we get the desired statement.

b) Let us make use of notations from heading a) and the result of a). Let

$$x = a_1^2 + b_1^2 + c_1^2 \quad \text{and} \quad y = a^2 + b^2 + c^2.$$

We have to verify that $x = y$. Further, let A_1, B_1 and C_1 be the intersection points of the medians of triangles DBC, DAC and DAB, respectively. The homothety with center D and coefficient $\frac{3}{2}$ sends the intersection point of the medians of triangle $A_1B_1C_1$ to the intersection point of the medians of triangle ABC. Therefore, M is the intersection point of the extension of median DX of tetrahedron $A_1B_1C_1D$ with the sphere circumscribed about this tetrahedron. Consequently, to compute the length of segment DM, we may make use of the formula for DN obtained in heading a):

$$DM = \frac{DA_1^2 + DB_1^2 + DC_1^2}{3DX}.$$

Clearly, $DX = \frac{2m}{3}$. Expressing DA_1, DB_1 and DC_1 in terms of medians and medians in terms of sides we get

$$DA_1^2 + DB_1^2 + DC_1^2 = \frac{4x - y}{9}.$$

Therefore, $DM = \frac{4x-y}{18m}$.

On the other hand, $DM = \frac{3}{4}m$; hence, $2(4x - y) = 27m^2$. By formula (1) we have $9m^2 = 3x - y$, hence, $2(4x - y) = 3(3x - y)$, i.e., $x = y$.

6.18. Let $CD = a$. Then $AC = \frac{a}{\sin\alpha}$, $BC = \frac{a}{\sin\beta}$ and $AB = a\sqrt{\cot^2\alpha + \cot^2\beta}$. We get the desired statement by taking into account that

$$AB^2 = AC^2 + BC^2 - 2AC \cdot BC \cos\varphi.$$

6.19. Let us consider the rectangular parallelepiped whose edges AB, AD and AA_1 are edges of the given tetrahedron. The segment that connects the midpoints of segments AB and A_1D is the parallel to midline BD_1 of triangle ABD_1; therefore, the length of this segment is equal to $\frac{1}{2}d$, where d is the length of the diagonal of the parallelepiped.

6.20. Since

$$S_{ABC}^2 = S_{ABD}^2 + S_{BCD}^2 + S_{ACD}^2$$

(see Problem 1.22), it follows that

$$S_{ABC} = \frac{\sqrt{a^2b^2 + b^2c^2 + a^2c^2}}{2}.$$

Therefore, the volume of tetrahedron is equal to

$$\frac{h\sqrt{a^2b^2 + b^2c^2 + a^2c^2}}{6}.$$

On the other hand, it is equal to $\frac{1}{6}abc$. By equating these expressions we get the desired statement.

6.21. On rays AC and AD, take points P and R so that $AP = AR = AB$ and consider square $APQR$. Clearly,

$$\triangle ABC = \triangle RQD \quad \text{and} \quad \triangle ABD = \triangle PQC;$$

hence, $\triangle BCD = \triangle QDC$. Thus, the sum of the plane angles at the vertex B is equal to

$$\angle PQC + \angle CQD + \angle DQR = \angle PQR = 90^\circ.$$

6.22. For each edge of tetrahedron there exists only one edge not neighbouring to it and, therefore, among any three edges there are two neighbouring ones. Now, notice that the three dihedral angles at edges of one face cannot be right ones. Therefore, two variants of the disposition of the three edges whose dihedral angles are right ones are possible:

1) These edges exit from one vertex;

2) Two edges exit from the endpoints of one edge.

In the first case it suffices to make use of the result of Problem 5.2.

Let us consider the second case: the dihedral angles at edges AB, BC and CD are right ones. Then tetrahedron $ABCD$ looks as follows: in triangles ABC and BCD angles ACB and CBD are right ones and the angle between the planes of these triangles is also a right one. In this case the angles ACB, ACD, ABD and CBD are right ones.

6.23. Thanks to the solution of Problem 6.22 the following two variants are possible.

1) All the plane angles at one vertex of the tetrahedron are right ones. But in this case the lengths of all the segments that connect midpoints of the opposite edges are equal (Problem 6.19).

2) The dihedral angles at edges AB, BC and CD are right ones. In this case edges AC and BD are perpendicular to faces CBD and ABC, respectively. Let $AC = 2x$, $BC = 2y$ and $BD = 2z$. Then the length of the segment that connects the midpoints of edges AB and CD as well as that of the segment that connects the midpoints of edges BC and AD is equal to $\sqrt{x^2 + z^2}$ and the length of the segment that connects the midpoints of edges AC and BD is equal to $\sqrt{x^2 + 4y^2 + z^2}$. Therefore,

$$x^2 + z^2 = a^2 \ \text{ and } \ x^2 + 4y^2 + z^2 = b^2.$$

The longest edge of tetrahedron $ABCD$ is AD; its squared length is equal to

$$4(x^2 + y^2 + z^2) = b^2 + 3a^2.$$

6.24. As follows from the solution of Problem 6.22, we may assume that the vertices of the given tetrahedron are the vertices A, B, D and D_1 of the rectangular parallelepiped $ABCDA_1B_1C_1D_1$. Let α be the angle to be found; $AB = a$, $AD = b$ and $DD_1 = c$. Then $a = b \tan \alpha$ and $c = b \tan \alpha$. The cosine of the angle between planes BB_1D and ABC_1 is equal to

$$\frac{ac}{\sqrt{a^2 + b^2}\sqrt{b^2 + c^2}} = \frac{\tan^2 \alpha}{1 + \tan^2 \alpha} = \sin^2 \alpha$$

(cf. Problem 1.9 a)). Therefore,

$$\cos \alpha = \sin^2 \alpha = 1 - \cos^2 \alpha,$$

i.e., $\cos\alpha = \frac{-1\pm\sqrt{5}}{2}$. Since $1+\sqrt{5} > 2$, we finally get $\alpha = \arccos(\frac{\sqrt{5}-1}{2})$.

6.25. a) Let $AB = CD$, $AC = BD$ and the sum of the plane angles at vertex A be equal to 180°. Let us prove that $AD = BC$. To this end it suffices to verify that $\angle ACD = \angle BAC$. But both the sum of the angles of triangle ACD and the sum of the plane angles at vertex A are equal to 180°; moreover, $\angle DAB = \angle ADC$ because $\triangle DAB = \triangle ADC$.

b) Let O_1 and O_2 be the tangent points of the inscribed sphere with faces ABC and BCD. Then $\triangle O_1BC = \triangle O_2BC$. The conditions of the problem imply that O_1 and O_2 are the centers of the circles circumscribed about the indicated faces. Hence,

$$\angle BAC = \frac{\angle BO_1C}{2} = \frac{\angle BO_2C}{2} = \angle BDC.$$

Similar arguments show that each of the plane angles at vertex D is equal to the corresponding angle of triangle ABC and, therefore, their sum is equal to 180°. This statement holds for all the vertices of the tetrahedron. It remains to make use of the result of Problem 2.32 a).

c) The angles ADB and ACB subtend equal chords in equal circles and, therefore, either they are equal or their sum is equal to 180°.

First, suppose that for each pair of angles of the faces of the tetrahedron that subtend the same edge the equality of angles takes place. Then, for example, the sum of the plane angles at vertex D is equal to the sum of angles of triangle ABC, i.e., is equal to 180°. The sum of the plane angles at any vertex of the tetrahedron is equal to 180° and, therefore, the tetrahedron is an equifaced one (see Problem 2.32 a)).

Now, let us prove that the case when the angles ADB and ACB are not equal is impossible. Suppose that $\angle ADB + \angle ACB = 180^\circ$ and $\angle ADB \neq \angle ACB$. Let, for definiteness, angle $\angle ADB$ be an obtuse one. It is possible to "unfold" the surface of tetrahedron $ABCD$ to plane ABC so that the images D_a, D_b and D_c of point D fall on the circle circumscribed about triangle ABC; in doing so we select the direction of the rotation of a lateral face about the edge in the base in accordance with the fact whether the angles that subtend this edge are equal (the positive direction) or their sum is equal to 180° (the negative direction).

In the process of unfolding point D moves along the circles whose planes are perpendicular to lines AB, BC and CA. These circles lie in distinct planes and, therefore, any two of them have not more than two common points. But each pair of these circles has two common points: point D and the point symmetric to it through plane ABC. Therefore, points D_a, D_b and D_c are pairwise distinct.

Moreover, $AD_b = AD_c$, $BD_a = BD_c$ and $CD_a = CD_b$. The unfolding now looks as follows: triangle AD_cB with obtuse angle D_c is inscribed in the circle; from points A and B chords AD_b and BD_a equal to AD_c and BD_c, respectively, are drawn; C is the midpoint of one of the two arcs determined by points D_a and D_b. One of the midpoints of these two arcs is symmetric to point D_c through the midperpendicular to segment AB; this point does not suit us.

The desired unfolding is depicted on Fig. 48. The angles at vertices D_a, D_b and D_c of the hexagon $AD_cBD_aCD_b$ complement the angles of triangle ABC to 180° and, therefore, their sum is equal to 360°. But these angles are equal to the plane angles at vertex D of tetrahedron $ABCD$ and, therefore, their sum is smaller than 360°. Contradiction.

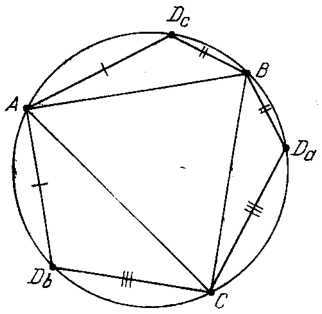

Figure 48 (Sol. 6.25)

d) Let K and L be the midpoints of edges AB and CD, let O be the center of mass of the tetrahedron, i.e., the midpoint of segment KL. Since O is the center of the circumscribed sphere of the tetrahedron, triangles AOB and COD are isosceles ones with equal lateral sides and equal medians OK and OL. Hence, $\triangle AOB = \triangle COD$ and, therefore, $AB = CD$.

The equality of the other pairs of opposite edges is similarly proved.

6.26. The trihedral angles at vertices A and C have equal dihedral angles and, therefore, they are equal (Problem 5.3). Consequently, their plane angles are also equal; hence, $\triangle ABC = \triangle CDA$.

6.27. The center of mass of the tetrahedron lies on the plane that connects the midpoints of edges AB and CD. Therefore, the center of the circumscribed sphere of the tetrahedron lies on this line, too; hence, the indicated plane is perpendicular to edges AB and CD. Let C' and D' be the projections of points C and D, respectively, to the plane passing through line AB parallel to CD. Since $AC'BD'$ is a parallelogram, it follows that $AC = BD$ and $AD = BC$.

6.28. Let K and L be the midpoints of edges AB and CD. The center of mass of the tetrahedron lies on line KL and, therefore, the center of the inscribed sphere also lies on line KL. Therefore, under the projection to the plane perpendicular to CD segment KL goes into the bisector of the triangle which is the projection of face ABC. It is also clear that the projection of point K is the midpoint of the projection of segment AB. Therefore, the projections of segments KL and AB are perpendicular, consequently, plane KDC is perpendicular to plane Π that passes through edge AB parallel to CD. Similarly, plane LAB is perpendicular to Π. Therefore, line KL is perpendicular to Π. Let C' and D' be the projections of points C and D, respectively, to plane Π. Since $AC'BD'$ is a parallelogram, $AC = BD$ and $AD = BC$.

6.29. Let S be the midpoint of edge BC; let K, L, M and N be the midpoints of edges AB, AC, DC and DB, respectively. Then $SKLMN$ is a tetrahedral angle with equal plane angles and its section $KLMN$ is a parallelogram. On the one hand, the tetrahedral angle with equal plane angles has a rhombus as a section (Problem 5.16 b)); on the other hand, any two sections of the tetrahedral angle which are parallelograms are parallel (Problem 5.16 a)).

Therefore, $KLMN$ is a rhombus; moreover, from the solution of Problem 5.16 b) it follows that $SK = SM$ and $SL = SN$. This means that $AB = DC$ and $AC = DB$. Therefore, $\triangle BAC = \triangle ABD$ and $BC = DB$.

6.30. The tangent point of the escribed sphere with plane ABC coincides with the projection H of point O_d (the center of the sphere) to plane ABC. Since the trihedral angle O_dABC is a right one, H is the intersection point of the heights of triangle ABC (cf. Problem 2.11).

Let O be the tangent point of the inscribed sphere with face ABC. From the result of Problem 5.13 b) it follows that the lines that connect points O and H with the vertices of triangle ABC are symmetric through its bisectors. It is not difficult to prove that this means that O is the center of the circle circumscribed about triangle ABC (it suffices to carry out the proof for an acute triangle because point H belongs to the face). Thus, the tangent point of the inscribed sphere with face ABC coincides with the center of the circumscribed circle of the face; for the other faces the proof of this fact is carried out similarly. It remains to make use of the result of Problem 6.25 b).

6.31. Let us complement the given tetrahedron to a rectangular parallelepiped (cf. Problem 6.48 a)); let x, y and z be the edges of this parallelpiped. Then

$$x^2 + y^2 = a^2,\ y^2 + z^2 = b^2 \text{ and } z^2 + x^2 = c^2.$$

Since $R = \frac{d}{2}$, where d is the diagonal of the parallelepiped and $d^2 = x^2 + y^2 + z^2$, it follows that

$$R^2 = \frac{x^2 + y^2 + z^2}{4} = \frac{a^2 + b^2 + c^2}{8}.$$

By adding up equalities $x^2 + y^2 = a^2$ and $z^2 + x^2 + c^2$ and subtracting from them the equality $y^2 + z^2 = b^2$ we get

$$x^2 = \frac{a^2 + c^2 - b^2}{2}.$$

We similarly find x^2 and z^2. Since the volume of the tetrahedron is one third of the volume of the parallelepiped (see the solution of Problem 3.4), we have

$$V^2 = \frac{(xyz)^2}{9} = \frac{(a^2 + b^2 - c^2)(a^2 + c^2 - b^2)(b^2 + c^2 - a^2)}{72}.$$

6.32. Let us complement the given tetrahedron to a rectangular parallelepiped (see Problem 6.48 a)). The intersection point of the bisector planes of the dihedral angles of the tetrahedron (i.e., the center of the inscribed ball) coincides with the center O of the parallelepiped. By considering the projections to the planes perpendicular to the edges of the tetrahedron it is easy to verify that the distance from the faces of the tetrahedron to the vertices of the parallelepiped distinct from the vertices of the tetrahedron is twice that from point O. Hence, these vertices are the centers of the escribed balls(spheres?). This proves both statements.

6.33. Let us complement the given tetrahedron to a rectangular parallelepiped. Let AA_1 be its diagonal, O its center. Point H_1 is the projection of point A_1 to face BCD (cf. Problem 2.11) and the center O_1 of the circumscribed circle of triangle BCD is the projection of point O. Since O is the midpoint of segment AA_1, points H and H_1 are symmetric through O_1.

Let us consider the projection of the parallelepiped to the plane perpendicular to BD, see Fig. 49; in what follows we make use of the notations from this figure rather than notations of the body in space(?). The height CC' of triangle BCD is parallel

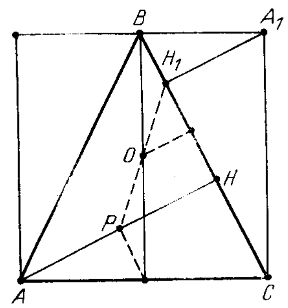

FIGURE 49 (SOL. 6.33)

to the plane of the projection and, therefore, the lengths of segments BH_1 and CH_1 are equal to h_1 and h_2; the lengths of segments AH and A_1H_1 do not vary under the projection. Since $AH : A_1H_1 = AC : A_1B = 2$ and $A_1H_1 : BH_1 = CH_1 : A_1H_1$, it follows that

$$AH^2 = 4H_1A_1^2 = 4h_1h_2.$$

6.34. Let us make use of the solution of the preceding problem and notations from Fig. 49. On this Figure, P is the midpoint of height AH. It is easy to verify that

$$OH = OH_1 = OP = \sqrt{r^2 + a^2},$$

where r is the distance from point O to the face and a the distance between the center of the circumscribed circle and the intersection point of the heights of the face.

6.35. a) Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_4$ be unit vectors perpendicular to the faces and directed outwards. Since the areas of all the faces are equal,

$$\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 = \mathbf{0}$$

(cf. Problem 7.19). Therefore,

$$0 = |\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4|^2 = 4 + 2\sum(\mathbf{e}_i, \mathbf{e}_j).$$

It remains to notice that the inner product $(\mathbf{e}_i, \mathbf{e}_j)$ is equal to $-\cos\varphi_{ij}$, where φ_{ij} is the dihedral angle between the i-th and j-th faces.

b) On one edge of the given trihedral angle with vertex S, take an arbitrary point A and draw from it segments AB and AC to the intersection with the other edges so that $\angle SAB = \angle ASC$ and $\angle SAC = \angle ASB$. Then $\triangle SCA = \triangle ABS$. Since the sum of the angles of triangle ACS is equal to the sum of plane angles at vertex S, it follows that $\angle SCA = \angle CSB$. Therefore, $\triangle SCA = \triangle CSB$; hence, tetrahedron $ABCS$ is an equifaced one. By heading a) the sum of the cosines of the dihedral angles at the edges of this tetrahedron is equal to 2 and this sum is twice the sum of the cosines of the dihedral angles of the given trihedral angle.

6.36. a) Let $AD \perp BC$. Then there exists plane Π passing through BC and perpendicular to AD. The height dropped from vertex B is perpendicular to AD and therefore, it lies in plane Π. Similarly, the height dropped from vertex C lies in plane Π. Therefore, these heights meet at a point. This point belongs also to plane Π' that passes through AD and is perpendicular to BC. It remains to notice that planes Π and Π' intersect along the common perpendicular to AD and BC.

b) Let heights BB' and CC' meet at one point. Each of the heights BB' and CC' is perpendicular to AD. Therefore, the plane that contains these heights is perpendicular to AD hence, $BC \perp AD$.

c) Let two pairs of opposite edges of the tetrahedron be perpendicular (to each other). Then the third pair of the opposite edges is also perpendicular (Problem 7.1).

Therefore, each pair of the tetrahedron's heights intersects. If several lines intersect pairwise, then either they lie in one plane or pass through one point. The heights of the tetrahedron cannot lie in one plane because otherwise all its vertices would lie in one plane; hence, they meet at one point.

6.37. From solution of Problem 6.36 a) it follows that the intersection point of the heights belongs to each common perpendicular to opposite pairs of edges.

6.38. a) Quadrilateral $KLMN$ is a parallelogram whose sides are parallel to AC and BD. Its diagonals, KM and LN, are equal if and only if it is a rectangle, i.e., $AC \perp BD$.

Notice also that plane $KLMN$ is perpendicular to the common perpendicular to AC and BD and divides it in halves.

b) Follows from the results of Problems 6.38 a) and 6.36 c).

6.39. a) Since $BC \perp AD$, there exists plane Π passing through line AD and perpendicular to BC; let U be the intersection point of line BC with plane Π. Then AU and DU are perpendiculars dropped from points A and D to line BC.

b) Let AU and DU be heights of triangles ABC and DBC. Then line BC is perpendicular to plane ADU and, therefore, $BC \perp AD$.

6.40. a) Follows from Problem 7.2.

b) Making use of the results of Problems 6.6 and 6.10 we see that the products of the cosines of the opposite dihedral angles are equal if and only if the sums of the squared lengths of the opposite edges are equal.

c) It suffices to verify that if all the angles between the opposite edges are equal to α, then $\alpha = 90^\circ$. Suppose that $\alpha \neq 90^\circ$, i.e., $\cos\alpha \neq 0$. Let a, b and c be the products of pairs of the opposite edges' lengths. One of the numbers $a\cos\alpha$, $b\cos\alpha$ and $c\cos\alpha$ is equal to the sum of the other two ones (Problem 6.51). Since $\cos\alpha \neq 0$, one of the numbers a, b and c is equal to the sum of the other two.

On the other hand, there exists a triangle the lengths of whose sides are equal to a, b and c (Problem 6.9). Contradiction.

6.41. a) If $ABCD$ is an orthocentrical tetrahedron, then

$$AB^2 + CD^2 = AD^2 + BC^2$$

(cf. Problem 6.40 a)). Therefore,

$$AB^2 + AC^2 - BC^2 = AD^2 + AC^2 - CD^2,$$

i.e., the cosines of angles BAC and DAC are of the same sign.

b) Since a triangle cannot have two nonacute angles, it follows that taking into account the result of heading a) we see that if $\angle BAC \geq 90^\circ$, then triangle BCD is an acute one.

6.42. Let K and L be the midpoints of edges AB and CD, respectively. Point H lies in the plane that passes through CD perpendicularly to AB and point O lies in the plane that passes through K perpendicularly to AB. These planes are symmetric through the center of mass of the tetrahedron, the midpoint M of segment KL. Consider such planes for all the edges; we see that points H and O are symmetric through M, hence, $KHLO$ is a parallelogram.

The squares of its sides are equal to $\frac{1}{4}(R^2 - AB^2)$ and $\frac{1}{4}(R^2 - CD^2)$; hence,

$$OH^2 = 2(R^2 - \frac{AB^2}{4}) + 2(R^2 - \frac{CD^2}{4}) - d^2 = 4R^2 - \frac{AB^2 + CD^2}{2} - d^2.$$

By considering the section that passes through M parallel to AB and CD we get $AB^2 + CD^2 = 4d^2$.

6.43. a) The circles of 9 points of triangles ABC and DBC belong to one sphere if and only if the bases of the heights dropped from vertices A and D to line BC coincide. It remains to make use of the result of Problem 6.39 b).

b) The segments that connect the midpoints of the opposite edges meet at one point that divides them in halves — the center of mass; moreover, for an orthocentric tetrahedron their lengths are equal (Problem 6.38 b)). Therefore, all the circles of 9 points of the tetrahedron's faces belong to the sphere whose diameter is equal to the length of the segment that connects the midpoints of the opposite edges and whose (sphere's) center coinsides with the tetrahedron's center of mass.

c) Both spheres pass through the midpoints of edges AB, BD, DC and CA and these points lie in the indicated plane.

6.44. Let O, M and H be the center of the circumscribed sphere, the center of mass and the intersection point of the heights of an orthocentric tetrahedron, respectively. It follows from the solution of Problem 6.42 that M is the midpoint of segment OH. The centers of mass of the tetrahedron's faces are the vertices of the tetrahedron homothetic to the given one with the center of homothety M and coefficient $-\frac{1}{3}$. Under this homothety point O goes to point O_1 that lies on segment MH and $MO_1 = \frac{1}{3}MO$. Therefore, $HO_1 = \frac{1}{3}HO$, i.e., the homothety with center H and coefficient $\frac{1}{3}$ sends point O into O_1. This homothety maps the vertices of the tetrahedron into the indicated points on the heights of the tetrahedron.

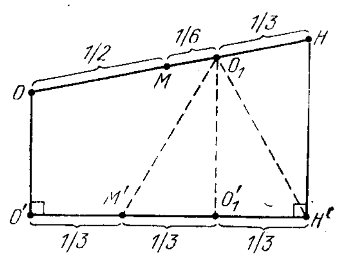

FIGURE 50 (SOL. 6.44)

Thus, 8 of 12 given points lie on the sphere of radius $\frac{1}{3}R$ centered at O_1, where R is the radius of the circumscribed sphere of the tetrahedron. It remains to show that the intersection points of the faces' heights also belong to the sphere. Let O',

H' and M' be the center of the circumscribed sphere, the intersection point of the heights and the center of mass of a face, respectively (Fig. 50). Point M' divides segment $O'H'$ in the ratio of $O'M' : M'H' = 1 : 2$ (see Plane, Problem 10.1).

Now, it is easy to calculate that the projection of point O_1 to the plane of this face coincides with the midpoint of segment $M'H'$ and, therefore, point O_1 is equidistant from M' and H'.

6.45. a) It follows from the solution of Problem 6.44 that under the homothety with center H and coefficient 3 point M' turns into a point on the circumscribed sphere of the tetrahedron.

b) It follows from the solution of Problem 6.44 that the homothety with center M and coefficient -3 maps point H' to a point on the circumscribed sphere of the tetrahedron.

6.46. Since $AB \perp CD$, there exists a plane passing through AB and perpendicular to CD. On this plane lie both Monge's point and the intersection point of the heights dropped from vertices A and B. If we draw such planes through all the edges, we see that they will have a unique common point.

6.47. Let us consider a tetrahedron in which the given segments connect the midpoints of the opposite edges and complement it to a parallelepiped. The edges of this parallelepiped are parallel to the given segments and its faces pass through the endpoints of these segments. Therefore, this parallelepiped is uniquely determined by the given segments and there are precisely two tetrahedrons that can be complemented to a given parallelepiped.

6.48. a) Two opposite edges of the tetrahedron serve as diagonals of the opposite faces of the obtained parallelepiped. These faces are rectangulars if and only if the opposite edges are equal.

The result of this heading is used in the solution of headings b)–d).

b) It suffices to notice that the given segments are parallel to the edges of the parallelepiped.

c) Let the areas of all the faces of the tetrahedron be equal. Let us complement tetrahedron AB_1CD_1 to the parallelepiped $ABCDA_1B_1C_1D_1$. Let us consider the projection to the plane perpendicular to line AC. Since the heights of triangles ACB_1 and ACD_1 are equal, the projection of triangle AB_1D_1 is an isosceles triangle and the projection of point A_1 is the midpoint of the base of the isosceles triangle. Therefore, edge AA_1 is perpendicular to face $ABCD$.

Similar arguments demonstrate that the parallelepiped is a rectangular one.

b) Let us make use of the notations of heading c) and consider again the projection to the plane perpendicular to AC. If the center of the inscribed sphere coincides with the center of mass, then plane ACA_1C_1 passes through the center of the inscribed sphere, i.e., is the bisector plane of the dihedral angle at edge AC. Therefore, the projection maps segment AA_1 to the bisector; hence, the median of the image under the projection of triangle AB_1D_1 is perpendicular to face $ABCD$ and so is edge AA_1.

6.49. Let us complement the equifaced tetrahedron to a parallelepiped. By Problem 6.48 a) we get a rectangular parallelepiped. If the edges of the parallelepiped are equal to a, b and c, then the squared lengths of the sides of the tetrahedron's face are equal to $a^2 + b^2, b^2 + c^2$ and $c^2 + a^2$. Since the sums of the squares of any two sides is greater than the square of the third side, the face is an acute triangle.

6.50. Let us complement the tetrahedron to a parallelepiped. The distances between the midpoints of the skew edges of the tetrahedron are equal to the lengths of the edges of this parallelepiped. It remains to make use of the fact that if a and b are the lengths of the sides of the parallelepiped and d_1 and d_2 are the lengths of its diagonals, then $d_1^2 + d_2^2 = 2(a^2 + b^2)$.

6.51. Let us complement the tetrahedron to a parallelepiped. Then a and a_1 are diagonals of the two opposite faces of the parallelepiped. Let m and n be the sides of these faces and $m \geq n$. By the law of cosines

$$4m^2 = a^2 + a_1^2 + 2aa_1 \cos\alpha; \quad 4n^2 = a^2 + a_1^2 - 2aa_1 \cos\alpha;$$

therefore,

$$aa_1 \cos\alpha = m^2 - n^2.$$

Write such equalities for numbers $bb_1 \cos\beta$ and $cc_1 \cos\gamma$ and compare; we get the desired statement.

6.52. Let us complement tetrahedron $ABCD$ to parallelepiped (Fig. 51). The section of this parallelepiped by plane Π is a parallelogram; points M and N lie on its sides and line l passes through the midpoints of the other two of its sides.

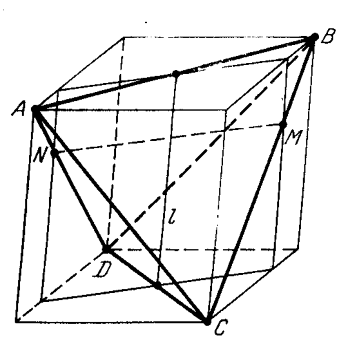

FIGURE 51 (SOL. 6.52)

6.53. Let AB_1CD_1 be the tetrahedron inscribed in cube $ABCDA_1BC_1D_1$; let H be the intersection point of diagonal AC_1 with plane B_1CD_1; let M be the midpoint of segment AH which serves as the tetrahedron's height. Since $C_1H : HA = 1 : 2$ (Problem 2.1), point M is symmetric to C_1 through plane B_1CD_1.

6.54. If α is the angle between the planes of any of the lateral faces and the plane of the base, h the height of the pyramid, then the distance from the projection of the vertex to the plane of the base from any other plane that contains an edge of the base is equal to $h \cot\alpha$.

Notice also that if there are equal dihedral angles at edges of the base not just angles between planes, then the projection of the vertex is the center of the inscribed circle.

6.55. Let h be the height of the pyramid, V its volume, S the area of the base. By Problem 6.54, $h = r \tan\alpha$, where r is the radius of the circle inscribed in the base. Hence,

$$V = \frac{Sh}{3} = \frac{Sr \tan\alpha}{3} = \frac{S^2 \tan\alpha}{3p} = \frac{(p-a)(p-b)(p-c)\tan\alpha}{3},$$

where $p = \frac{1}{2}(a+b+c)$.

6.56. Let line AM intersect BC at point P. Then

$$MA_1 : SA = MP : AP = S_{MBC} : S_{ABC}.$$

Similarly,

$$MB_1 : SB = S_{AMC} : S_{ABC} \text{ and } MC_1 : SC = S_{ABM} : S_{ABC}.$$

By adding up these equalities and taking into account that

$$S_{MBC} + S_{AMC} + S_{ABM} = S_{ABC}$$

we get the desired statement.

6.57. Let O be the center of the base of the cone. In the trihedral angles $SBOC$, $SCOA$ and $SAOB$, the dihedral angles at edges SB and SC, SC and SA, SA and SB, respectively, are equal. Denote these angles by x, y and z. Then $\alpha = y + z$, $\beta = z + x$ and $\gamma = x + y$. Since plane SCO is perpendicular to the plane tangent to the surface of the cone along the generator SC, the angle to be found is equal to

$$\frac{\pi}{2} - x = \frac{\pi + \alpha - \beta - \gamma}{2}.$$

6.58. a) Let us drop from M perpendicular MO to plane ABC. Since the distance from point A_1 to plane ABC is equal to the distance from point A to plane BC, the angle between the planes ABC and A_1BC is equal to 45°. Therefore, the distance from point O to line BC is equal to the length of segment MO. Similarly, the distances from point O to lines CA and AB are equal to the length of segment MO and, therefore, O is the center of the inscribed circle of triangle ABC and $MO = r$.

b) Let P be the intersection point of lines B_1C and BC_1. Then planes AB_1C and ABC_1 intersect along line AP and planes A_1BC_1 and A_1B_1C intersect along line A_1P. Similar arguments show that the projection of point N to plane ABC coincides with the projection of point M, i.e., it is the center O of the circle inscribed in triangle ABC.

First solution. Let h_a, h_b and h_c be the heights of triangle ABC; Q the projection of point P to plane ABC. By considering trapezoid BB_1C_1C we deduce that $PQ = \frac{h_bh_c}{h_b+h_c}$. Since

$$AO : OQ = AB : BQ = (b+c) : a,$$

it follows that

$$NO = \frac{aAA_1 + (b+c)PQ}{a+b+c} = \frac{ah_a(h_b+h_c) + (b+c)h_bh_c}{(a+b+c)(h_b+h_c)} = \frac{4S}{a+b+c} = 2r.$$

Second solution. Let K be the intersection point of line NO with plane $A_1B_1C_1$. From the solution of Problem 3.20 it follows that $MO = \frac{1}{3}KO$ and $NK = \frac{1}{3}KO$; hence, $NO = 2MO = 2r$.

6.59. Let p and q be the lengths of the sides of the bases of the pyramid. Then the area of the lateral face is equal to $\frac{1}{2}a(p+q)$. Let us consider the section of the pyramid by the plane that passes through the center of the inscribed ball perpendicularly to one of the sides of the base. This section is a circumscribed trapezoid with lateral side a and bases p and q. Therefore, $p+q=2a$. Hence, the area of the lateral side of the pyramid is equal to $4a^2$.

6.60. Let N_i be the base of the perpendicular dropped from point M to the edge of the base (or its extension) so that M_i lies in the plane of the face that passes through this edge. Then

$$MM_i = N_iM\tan\alpha,$$

where α is the angle between the base and the lateral face of the pyramid. Therefore, we have to prove that the sum of lengths of segments N_iM does not depend on point M. Let us divide the base of the pyramid into triangles by segments that connect point M with vertices. The sum of the areas of these triangles is equal to

$$\frac{a}{2}N_1M+\dots+\frac{a}{2}N_nM,$$

where a is the length of the edge at the base of the pyramid. On the other hand, the sum of the areas of these triangles is always equal to the area of the base.

6.61. If the sphere is tangent to the sides of the dihedral angle, then, after the identification of these sides, the tangent points coincide. Therefore, all the tangent points of the lateral faces with the inscribed sphere go under rotations about edges into the same point — the tangent point of the sphere with the plane of the pyramid's base.

The distances from this point to the vertices of faces (after rotations) are equal to the distances from the tangent points of the sphere with the lateral faces to the vertex of the pyramid. It remains to notice that the lengths of all the tangents to the sphere dropped from a vertex of the pyramid are equal.

6.62. Let us prove that all the lines indicated are parallel to the plane tangent to the circumscribed sphere of the pyramid at its vertex. To this end it suffices to verify that if AA_1 and BB_1 are the heights of triangle ABC, then line A_1B_1 is parallel to the line tangent to the circumscribed circle of the triangle at point C. Since

$$A_1C : B_1C = AC\cos C : BC\cos C = AC : BC,$$

it follows that $\triangle A_1B_1C \sim \triangle ABC$. Therefore, $\angle CA_1B_1 = \angle A$. It is also clear that the angle between the tangent to the circumscribed circle at point C and chord BC is equal to $\angle A$.

6.63. First, let us suppose that the lateral edges of the pyramid form equal angles with the indicated ray SO. Let the plane perpendicular to ray SO intersect the lateral edges of the pyramid at points A_1, B_1, C_1 and D_1. Since $SA_1 = SB_1 = SC_1 = SD_1$ and the areas of triangles BCD, ADB, ABC and ACD are equal, it follows that making use of the result of Problem 3.37 we get the desired statement.

Now, suppose that $SA+SC=SB+SD$. On the lateral edges of the pyramid draw equal segments SA_1, SB_1, SC_1 and SD_1. Making use of the result of Problem 3.37 it is easy to deduce that points A_1, B_1, C_1 and D_1 lie in one plane Π. Let S_1 be the circumscribed circle of triangle $A_1B_1C_1$, O its center, i.e., the projection of vertex S to plane Π. Point D_1 lies in plane Π and the distance from it to vertex S is equal to the distance from points on circle S_1 to vertex S. Therefore, point D_1 lies on the circumscribed circle of triangle $A_1B_1C_1$, i.e., ray SO is the desired one.

6.64. Let line l intersect line AB_1 at point K. The statement of the problem is equivalent to the fact that planes KBC_1, KCD_1 and KDA_1 have a common line, in particular, they have a common point distinct from K. Let us draw a plane parallel to the bases of the pyramid through point K. Let L, M and N be the intersection points of this plane with lines BC_1, CD_1 and DA_1, see Fig. 52 a); let $A_0B_0C_0D_0$ be the parallelogram along which this plane intersects the given pyramid or the extensions of its edges. Points K, L, M and N divide the sides of the parallelogram $A_0B_0C_0D_0$ in the same ratio, i.e., $KLMN$ is also a parallelogram. Planes KBC_1 and KDA_1 intersect plane $ABCD$ along the lines that pass through points B, C and D, respectively, parallel to lines KL, KM and KN, respectively. It remains to prove that these three lines meet at one point.

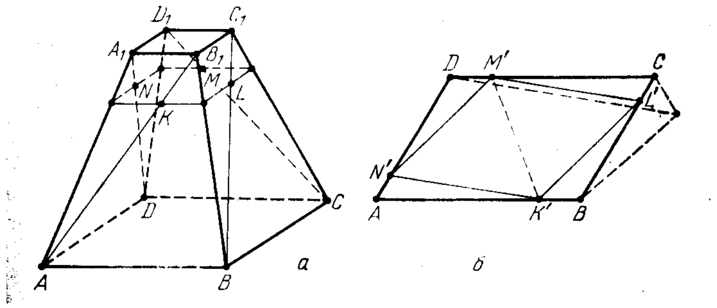

FIGURE 52 (SOL. 6.64)

On sides of parallelogram $ABCD$, take points K', L', M' and N' that divide these sides in the same ratio in which points K, L, M and N divide the sides of parallelogram $A_0B_0C_0D_0$. We have to prove that lines passing through points B, C and D parallel to lines $K'L'$, $K'M'$ and $L'M'$, respectivley, meet at one point (Fig. 52 b)).

Notice that the lines passing through vertices K', L' and M' of triangle $K'L'M'$ parallel to lines BC, BD and CD intersect at point M symmetric to point M' through the midpoint of segment CD. Therefore, the lines passing through points B, C and D parallel to lines $K'L'$, $K'M'$ and $L'M'$, respectively, also meet at one point (see \$).

REMARK. Since a linear transformation makes the parallelogram $ABCD$ into a square, it suffices to prove the required statement for a square. If $ABCD$ is a square, then $K'L'M'N'$ is also a square. It is easy to verify that the lines that pass through points B, C and D parallel to lines $K'L'$, $K'M'$ and $K'N'$, respectively, meet at one point that lies on the circumscribed circle of the square $ABCD$.

6.65. If p is the semiperimeter of the base of the prism, r the radius of the sphere, then the area of the base is equal to pr and the area of the lateral surface is equal to $4pr$. Therefore, the total surface area of the prism is equal to $6S$.

6.66. a) Let M and N be the midpoints of edges PP_1 and AA_1. Clearly, tetrahedron AA_1PP_1 is symmetric through line MN. Further, let P' be the projection of point P to the plane of face ACC_1A_1. Point P' lies on the projection $B'B_1'$ of segment BB_1 to this plane and divides it in the ratio of $B'P' : P'B_1' = 1 : 2$. Therefore, P' is the midpoint of segment AP_1. Therefore, planes APP_1 and AA_1P_1 are perpendicular to each other. Similarly, planes A_1PP_1 and AA_1P are perpendicular.

b) Since PP_1N is the bisector plane of the dihedral angle at edge PP_1 of the tetrahedron AA_1PP_1, it suffices to verify that the sum of the dihedral angles at edges PP_1 and AP of tetrahedron APP_1N is equal to 90°.

Plane PP_1N is perpendicular to face BCC_1B_1, therefore, we have to verify that the angle between planes PP_1A and BCC_1B_1 is equal to the angle between planes PP_1A and ABB_1A_1. These angles are equal because under the symmetry through line PP' plane PP_1A turns into itself and the indicated planes of the faces turn into each other.

CHAPTER 7. VECTORS AND GEOMETRIC TRANSFORMATIONS

§1. Inner (scalar) product. Relations

7.1. a) Given a tetrahedron $ABCD$, prove that

$$(\{AB\},\{CD\})+(\{AC\},\{DB\})+(\{AD\},\{BC\})=0.$$

b) In a tetrahedron, prove that if two pairs of opposite edges are perpendicular, then the third pair of opposite edges is also perpendicular.

7.2. Prove that the sum of squared lengths of two opposite pairs of a tetrahedron's edges are equal if and only if the third pair of opposite edges is perpendicular.

7.3. The diagonal AC_1 of rectangular parallelepiped $ABCDA_1B_1C_1D_1$ is perpendicular to plane A_1BD. Prove that this parallelepiped is a cube.

7.4. In a regular truncated pyramid, point K is the midpoint of side AB of the upper base, L is the midpoint of side CD of the lower base. Prove that the lengths of projections of segments AB and CD to line KL are equal.

7.5. Given a trihedral angle with vertex S, point N, and a sphere that, passing through points S and N, intersects the edges of the trihedral angle at points A, B and C. Prove that the centers of mass of triangles ABC for various spheres belong to one plane.

7.6. Prove that the sum of the distances from an inner point of a convex polyhedron to the planes of its faces does not depend on the position of the point if and only if the sum of the outer unit vectors perpendicular to the faces faces of the polyhedron is equal to zero.

7.7. Prove that in an orthocentric tetrahedron the center of mass is the midpoint of the segment that connects the orthocenter with the center of the circumscribed sphere.

§2. Inner product. Inequalities

7.8. Prove that it is impossible to select more than 4 vectors in space all the angles between which are obtuse ones.

7.9. Prove that it is impossible to select more than 6 vectors in space all the angles between which are not acute ones.

7.10. Prove that the sum of the cosines of the dihedral angles in a tetrahedron is positive and does not exceed 2.

7.11. Inside a convex polyhedron $A_1\dots A_n$, a point A is taken and inside a convex polyhedron $B_1\dots B_n$ a point B is taken. Prove that if $\angle A_iAA_j\leq\angle B_iBB_j$ for all i, j, then all these inequalities are, actually, equalities.

§3. Linear dependence of vectors

7.12. Points O, A, B and C do not lie in one plane. Prove that point X lies in plane ABC if and only if

$$\{OX\}=p\{OA\}+q\{OB\}+r\{OC\},$$

where
$$p+q+r=1.$$
Moreover, if point X belongs to triangle ABC, then
$$p:q:r=S_{BXC}:S_{CXA}:S_{AXB}.$$

7.13. On edges AB, AC and AD of tetrahedron $ABCD$, points K, L and M are fixed. We have $AB=\alpha AK$, $AG=\beta AL$ and $AD=\gamma AM$.

a) Prove that if
$$\gamma=\alpha+\beta+1,$$
then all planes KLM contain a fixed point.

b) Prove that if
$$\beta=\alpha+1 \text{ and } \gamma=\beta+1,$$
then all the planes KLM contain a fixed line.

7.14. Two regular pentagons $OABCD$ and $OA_1B_1C_1D_1$ with common vertex O do not lie in one plane. Prove that lines AA_1, BB_1, CC_1 and DD_1 are parallel to one plane.

7.15. a) Inside tetrahedron $ABCD$ a point O is taken. Prove that if
$$\alpha\{OA\}+\beta\{OB\}+\gamma\{OC\}+\delta\{OD\}=\{0\},$$
then all the numbers α, β, γ and δ are of the same sign.

b) From point O inside a tetrahedron perpendiculars $\{OA_1\}$, $\{OB_1\}$, $\{OC_1\}$ and $\{OD_1\}$ are dropped to the tetrahedron's faces. Prove that if
$$\alpha\{OA_1\}+\beta\{OB_1\}+\gamma\{OC_1\}+\delta\{OD_1\}=\{0\},$$
then all the numbers α, β, γ and δ are of the same sign.

7.16. Point O lies inside polyhedron $A_1\dots A_n$. Prove that there exist positive (and, therefore, nonzero) numbers $x_1, \dots, x_n$ such that
$$x_1\{OA_1\}+\dots+x_n\{OA_n\}=\{0\}.$$

§4. Miscellaneous problems

7.17. Let **a**, **b**, **c** and **d** be unit vectors directed from the center of a regular tetrahedron to its vertices and **u** an arbitrary vector. Prove that
$$(\mathbf{a},\mathbf{u})\mathbf{a}+(\mathbf{b},\mathbf{u})\mathbf{b}+(\mathbf{c},\mathbf{u})\mathbf{c}+(\mathbf{d},\mathbf{u})\mathbf{d}=\frac{4}{3}\mathbf{u}.$$

7.18. From point M inside a regular tetrahedron perpendiculars MA_i ($i=1$, 2, 3, 4) are dropped to its faces. Prove that
$$\{MA_1\}+\{MA_2\}+\{MA_3\}+\{MA_4\}=\frac{4}{3}\{MO\},$$
where O is the center of the tetrahedron.

7.19. From a point O inside a convex polyhedron rays that intersect the planes of the polyhedron's faces and perpendicular to them are drawn. On these rays, vectors are drawn from point O, the lengths of these vectors measured in chosen linear units are equal to the areas of the corresponding faces measured in the corresponding area units. Prove that the sum of these vectors is equal to zero.

7.20. Given three pairwise perpendicular lines the distance between any two of which is equal to a. Find the volume of the parallelepiped whose diagonal lies on one of the lines and diagonals of two neighbouring faces on the two other lines.

7.21. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be arbitrary vectors. Prove that

$$|\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| + |\mathbf{a}+\mathbf{b}+\mathbf{c}| \geq |\mathbf{a}+\mathbf{b}| + |\mathbf{b}+\mathbf{c}| + |\mathbf{c}+\mathbf{a}|.$$

§5. Vector product

The *vector product* of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the vector $\mathbf{c}$ whose length measured in chosen linear units is equal to the area of the parallelogram formed by vectors $\mathbf{a}$ and $\mathbf{b}$ measured in the corresponding area units, which is perpendicular to $\mathbf{a}$ and $\mathbf{b}$, and which is directed in such a way that the triple $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is a "right" one.

Recall that the triple of vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is a "right" one if the orientation of the triple is the same as that of a thumb ($\mathbf{a}$), index finger ($\mathbf{b}$) and the middle finger ($\mathbf{c}$) of the right hand. Notation: $\mathbf{c} = [\mathbf{a}, \mathbf{b}]$; another notation: $\mathbf{c} = \mathbf{a} \times \mathbf{b}$.

7.22. Prove that

a) $[\mathbf{a}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}]$;

b) $[\lambda\mathbf{a}, \mu\mathbf{b}] = \lambda[\mathbf{a}, \mathbf{b}]$;

c) $[\mathbf{a}, \mathbf{b}+\mathbf{c}] = [\mathbf{a}, \mathbf{b}] + [\mathbf{a}, \mathbf{c}]$.

7.23. The coordinates of vectors $\mathbf{a}$ and $\mathbf{b}$ are (a_1, a_2, a_3) and (b_1, b_2, b_3). Prove that the coordinates of $[\mathbf{a}, \mathbf{b}]$ are

$$(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

7.24. Prove that

a) $[\mathbf{a}, [\mathbf{b}, \mathbf{c}]] = \mathbf{b}(\mathbf{a}, \mathbf{c}) - \mathbf{c}(\mathbf{a}, \mathbf{b})$;

b) $([\mathbf{a}, \mathbf{b}], [\mathbf{c}, \mathbf{d}]) = (\mathbf{a}, \mathbf{c})(\mathbf{b}, \mathbf{d}) - (\mathbf{b}, \mathbf{c})(\mathbf{a}, \mathbf{d})$.

7.25. a) Prove that (the Jacobi identity):

$$[\mathbf{a}, [\mathbf{b}, \mathbf{c}]] + [\mathbf{b}, [\mathbf{c}, \mathbf{a}]] + [\mathbf{c}, [\mathbf{a}, \mathbf{b}]] = \mathbf{0}.$$

b) Let point O lie inside triangle ABC and $\mathbf{a} = \{OA\}$, $\mathbf{b} = \{OB\}$ and $\mathbf{c} = \{OC\}$. Prove that the Jacobi identity for vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is equivalent to the identity

$$\mathbf{a}S_{BOC} + \mathbf{b}S_{COA} + \mathbf{c}S_{OAB} = \mathbf{0}.$$

7.26. The angles at the vertices of a spatial hexagon are right ones and the hexagon has no parallel sides. Prove that the common perpendiculars to the pairs of the opposite sides of the hexagon are perpendicular to one line.

7.27. Prove with the help of vector product the statement of Problem 7.19 for tetrahedron $ABCD$.

7.28. a) Prove that the planes passing through the bisectors of the faces of trihedral angle $SABC$ perpendicularly to the planes of these faces intersect along one line and this line is determined by the vector

$$[\mathbf{a}, \mathbf{b}] + [\mathbf{b}, \mathbf{c}] + [\mathbf{c}, \mathbf{a}],$$

where **a**, **b** and **c** are unit vectors directed along edges SA, SB and SC, respectively.

b) On the edges of a trihedral angle with vertex O points A_1, A_2 and A_3 are taken (one on each edge) so that $OA_1 = OA_2 = OA_3$. Prove that the bisector planes of its dihedral angles intersect along one line determined by the vector

$$\{OA_1\}\sin\alpha_1 + \{OA_2\}\sin\alpha_2 + \{OA_3\}\sin\alpha_3,$$

where α_i is the value of the plane angle opposite to edge OA_i.

7.29. Given parallelepiped $ABCDA_1B_1C_1D_1$, prove that the sum of squares of the areas of three of its pairwise nonparallel faces is equal to the sum of squares of areas of faces of the tetrahedron A_1BC_1D.

The number $([\mathbf{a}, \mathbf{b}], \mathbf{c})$ is called the *mixed product* of vectors **a**, **b** and **c**. It is easy to verify that the absolute value of this number is equal to the volume of the parallelepiped formed by vectors **a**, **b** and **c** and this number is positive if **a**, **b** and **c** is a right triple of vectors and negative otherwise.

7.30. Prove that vectors with coordinates (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_1, c_2, c_3) are parallel to one plane if and only if

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 = a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1.$$

Remark. For those acquainted with the notion of the product of matrices we can elucidate the relation between the vector product and the commutator of two matrices. To every vector $\mathbf{a} = (a_1, a_2, a_3)$ in three-dimensional space we can assign the skew-symmetric matrix

$$A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}.$$

Let matrices A and B be assigned to vectors **a** and **b**. Consider the matrix $[A, B] = AB - BA$, the commutator of matrices A and B. Easy calculations demonstrate that the matrix $[A, B]$ corresponds to the vector $[\mathbf{a}, \mathbf{b}]$.

§6. Symmetry

The *symmetry through point* A is the transformation of the space that sends point X into point X' such that A is the midpoint of segment XX'. Other names for this transformation are the *central symmetry with center* A or just the *symmetry with center* A.

7.31. Given a tetrahedron and point N, through every edge of the tetrahedron a plane is drawn parallel to the segment that connects point N with the midpoint of the opposite edge. Prove that all these six planes intersect at one point.

7.32. a) Through the midpoint of each edge of a tetrahedron the plane perpendicular to the opposite edge is drawn. Prove that all the six such planes intersect at one point. (*Monge's point.*)

b) Prove that if Monge's point lies in the plane of a face of the tetrahedron, then the base of the height dropped to this face lies on the circle circumscribed about this face.

The *symmetry through plane* Π is a transformation of the space that sends point X to point X' such that plane Π passes through the midpoint of segment XX' perpendicularly to it.

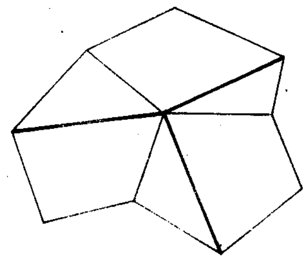

FIGURE 53 (7.33)

7.33. Three equal right pentagons are situated in space so that they have a common vertex and every two of them have a common edge. Prove that segments depicted on Fig. 53 by solid lines are the edges of a right trihedral angle.

7.34. Given two intersecting planes and a sphere tangent to them. All the spheres tangent to these planes and the given sphere are considered. Find the locus of the tangent points of these spheres.

7.35. Let O be the center of the cylinder (i.e., the midpoint of its axis), AB a diameter of one of the bases, C the point on the circle of the other base. Prove that the sum of dihedral angles of the trihedral angle $OABC$ with vertex O is equal to 2π.

7.36. In a convex pentahedral pyramid $SABCDE$, the lateral edges are equal and the dihedral angles at the lateral edges are equal. Prove that this pyramid is a regular one.

7.37. What maximal number of planes of symmetry a spatial figure consisting of three pairwise nonparallel lines can have?

The *symmetry through line* l is a transformation of the space that sends point X to a point X' such that line l passes through the midpoint of segment XX' perpendicularly to it. This transformation is also called the *axial symmetry* and l the *axis of the symmetry.*

7.38. Prove that symmetry through the line determined by vector $\mathbf{b}$ sends vector $\mathbf{a}$ to vector

$$2\mathbf{b}\frac{(\mathbf{a},\mathbf{b})}{(\mathbf{b},\mathbf{b})} - \mathbf{a}.$$

7.39. Perpendicular lines l_1 and l_2 intersect at one point. Prove that the composition of symmetries through these lines is a symmetry through the line perpendicular to both of them.

7.40. Prove that no body in space can have a nonzero even number of axes of symmetry.

§7. Homothety

Fix point O in space and number k. A *homothety* is the transformation of the space that sends point X to point X' such that $\{OX'\} = k\{OX\}$ Point O is called the *center of the homothety* and k the *coefficient of homothety.*

7.41. Let r and R be the radii of the inscribed and circumscribed spheres of a tetrahedron. Prove that $R \geq 3r$.

7.42. In the plane of a lateral face of a regular quadrilateral pyramid an arbitrary figure Φ is taken. Let Φ_1 be the projection of Φ to the base of the pyramid and Φ_2 the projection of Φ_1 to a lateral face adjacent to the initial one. Prove that figures Φ and Φ_2 are similar.

7.43. Prove that inside any convex polyhedron M two polyhedrons similar to it with coefficient $\frac{1}{2}$ can be placed so that they do not intersect.

7.44. Prove that a convex polyhedron cannot be covered with three polyhedrons homothetic to it with coefficient k, where $0 < k < 1$.

7.45. Given triangle ABC in plane, find the locus of points D in space such that segment OM, where O is the center of the sphere circumscribed about tetrahedron ABC and M is the center of mass of this tetrahedron, is perpendicular to plane ADM.

§8. Rotation. Compositions of transformations

We will not give a rigorous definition of a rotation about line l. For the solution of the problems to follow it suffices to have the following idea about a rotation: a *rotation about line* l (or *about axis* l) through an angle of φ is a transformation of the space that sends every plane Π perpendicular to l into itself and in Π this transformation is a rotation with center O through an angle of φ, where O is the intersection point of Π with l. In other words, under the rotation through an angle of φ about l point X turns into a point X' such that:

a) perpendiculars dropped from points X and X' to l have a common base O;

b) $OX = OX'$;

c) the angle of rotation from vector $\{OX\}$ to vector $\{OX'\}$ is equal to φ.

7.46. Let A_i' and A_i'' be the projections of the vertices of tetrahedron $A_1A_2A_3A_4$ to planes Π' and Π''. Prove that one of these planes can be moved in space so that the four lines $A_i'A_i''$ becomes parallel.

The *composition* of transformations F and G is the transformation $G \circ F$ that sends point X to point $G(F(X))$. Observe that, generally, $G \circ F \neq F \circ G$.

7.47. Prove that the composition of symmetries through two planes that intersect along line l is a rotation about l and the angle of this rotation is twice the angle of the rotation about l that sends the first plane into the second one.

7.48. Prove that the composition of the symmetry through point O with the rotation about line l passing through O is equal to the composition of a rotation about l and the symmetry through plane Π passing through point O perpendicularly to l.

A *motion* of space is a transformation of space such that if A' and B' are the images of points A and B, then $AB = A'B'$. In other words, a motion is a transformation of the space that preserves distances.

One can show that a motion that preserves four points in space not in one plane preserves the other points of the space as well. Therefore, any motion is given by the images of any four points not in one plane.

7.49. a) Prove that any motion of space is the composition of not more than four symmetries through planes.

b) Prove that any motion of space with a fixed point O is the composition of not more than three symmetries through planes.

A motion which is the composition of an even number of symmetries through planes is called a motion of *the first kind* or a motion that *preserves orientation* of the space. A motion which is the composition of an odd number of symmetries through planes is called a motion of *the second kind* or a motion that *changes the orientation* of the space.

We will not prove that the composition of an even number of symmetries with respect to planes cannot be represented in the form of the composition of an odd number of symmetries with respect to planes (though this is true).

7.50. a) Prove that any motion of the first kind with the fixed point is a rotation through an axis.

b) Prove that any motion of the second kind with the fixed point is the composition of a rotation through an axis (perhaps, through the zero angle) and the symmetry through a plane perpendicular to this axis.

7.51. A ball that lies in a corner of a parallelepipedal box rolls along the bottom of the box into another corner so that it is one and the same point on the ball that always touches the wall. From the second corner the ball rolls to the third one, then to the fourth one and, finally, returns to the initial corner. As a result, point X on the surface of the ball turns into point X_1. After similar rolling, point X_1 turns into X_2 and X_2 turns into X_3. Prove that points X, X_1, X_2 and X_3 lie in one plane.

§9. Reflexion of the rays of light

7.52. A ray of light enters a right trihedral angle, is reflected from all the faces once and then exits the trihedral angle. Prove that when the ray exits it goes along the line parallel to the line it entered the trihedral angle but in the opposite direction.

7.53. A ray of light falls on a flat mirror under an angle of α. The mirror is rotated through an angle of β about the projection of the ray to the mirror. Through which angle will the reflected ray move after the rotation of the mirror?

7.54. Plane Π passes through the vertex of a cone perpendicularly to its axis; point A lies in plane Π. Let M be a point of the cone such that the ray of light that goes from A to M becomes parallel to plane Π after being reflected from the surface of the cone as from the mirror. Find the locus of projections of points M to plane Π.

Problems for independent study

7.55. Point X lies at distance d from the center of a regular tetrahedron. Prove that the sum of squared distances from point X to the vertices of the tetrahedron is equal to $4(R^2 + d^2)$, where R is the radius of the circumscribed sphere of the tetrahedron.

7.56. On edges DA, DB and DC of tetrahedron $ABCD$ points A_1, B_1 and C_1, respectively, are taken so that $DA_1 = \alpha DA$, $DB_1 = \beta DB$ and $DC_1 = \gamma DC$. In which ratio plane $A_1B_1C_1$ divides segment DD', where D' is the intersection point of the medians of face ABC?

7.57. Let M and N be the midpoints of edges AB and CD of tetrahedron $ABCD$. Prove that the midpoints of segments AN, CM, BN and DM are the vertices of a parallelogram.

7.58. Let O be the center of the sphere circumscribed about an orthocentric tetrahedron, H its orthocenter. Prove that

$$\{OH\} = \frac{1}{2}(\{OA\} + \{OB\} + \{OC\} + \{OD\}).$$

7.59. Point X lies inside a regular tetrahedron $ABCD$ with center O. Prove that among the angles with vertex at point X that subtend the edges of the tetrahedron there is an angle whose value is not less than that of angle $\angle AOB$ and an angle whose value is not greater than that of angle $\angle AOB$.

Solutions

7.1. a) Let $\mathbf{a} = \{AB\}$, $\mathbf{b} = \{BC\}$, $\mathbf{c} = \{CD\}$. Then

$$(\{AB\}, \{CD\}) = (\mathbf{a}, \mathbf{c}),$$
$$(\{AC\}, \{DB\}) = (\mathbf{a} + \mathbf{b}, -\mathbf{b} - \mathbf{c}) = -(\mathbf{a}, \mathbf{b}) - (\mathbf{b}, \mathbf{b}) - (\mathbf{b}, \mathbf{c}) - (\mathbf{a}, \mathbf{c}),$$
$$(\{AD\}, \{BC\}) = (\mathbf{a} + \mathbf{b} + \mathbf{c}, \mathbf{b}) = (\mathbf{a}, \mathbf{b}) + (\mathbf{b}, \mathbf{b}) + (\mathbf{c}, \mathbf{b}).$$

Adding up these equalities we get the desired statement.

b) Follows obviously from heading a).

7.2. Let $\mathbf{a} = \{AB\}$, $\mathbf{b} = \{BC\}$ and $\mathbf{c} = \{CD\}$. The equality

$$AC^2 + BD^2 = BC^2 + AD^2$$

means that

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{b} + \mathbf{c}|^2 = |\mathbf{b}|^2 + |\mathbf{a} + \mathbf{b} + \mathbf{c}|^2,$$

i.e., $(\mathbf{a}, \mathbf{c}) = 0$.

7.3. Let $\mathbf{a} = \{AA_1\}$, $\mathbf{b} = \{AB\}$ and $\mathbf{c} = \{AD\}$. Then $\{AC_1\} = \mathbf{a} + \mathbf{b} + \mathbf{c}$ and, therefore, vector $\mathbf{a} + \mathbf{b} + \mathbf{c}$ is perpendicular to vectors $\mathbf{a} - \mathbf{b}$, $\mathbf{b} - \mathbf{c}$ and $\mathbf{c} - \mathbf{a}$ by the hypothesis. Taking into account that $(\mathbf{a}, \mathbf{b}) = (\mathbf{b}, \mathbf{c}) = (\mathbf{c}, \mathbf{a}) = 0$ we get

$$0 = (\mathbf{a} + \mathbf{b} + \mathbf{c}, \mathbf{a} - \mathbf{b}) = a^2 - b^2.$$

Similarly, $b^2 = c^2$ and $c^2 = a^2$. Therefore, the lengths of all the edges of the given rectangular parallelepiped are equal, i.e., this parallelepiped is a cube.

7.4. If vector $\mathbf{z}$ lies in the plane of the upper (or lower) base, then we will denote by $R\mathbf{z}$ the vector obtained from $\mathbf{z}$ by rotation through an angle of 90° (in that plane) in the positive direction. Let O_1 and O_2 be the centers of the upper and lower bases; $\{O_1K\} = \mathbf{a}$ and $\{O_1L\} = \mathbf{b}$. Then $\{AB\} = kR\mathbf{a}$ and $\{CD\} = kR\mathbf{b}$. We have to verify that $|(\{KL\}, \{AB\})| = |(\{KL\}, \{CD\})|$, i.e., $|(\mathbf{b} - \mathbf{a} + \mathbf{c}, kR\mathbf{a})| = |(\mathbf{b} - \mathbf{a} + \mathbf{c}, kR\mathbf{b})|$, where $\mathbf{c} = \{O_1O_2\}$. Taking into account that the inner product of perpendicular vectors is equal to zero we get

$$(\mathbf{b} - \mathbf{a} + \mathbf{c}, kR\mathbf{a}) = k(\mathbf{b}, R\mathbf{a}) \quad \text{and} \quad (\mathbf{b} - \mathbf{a} + \mathbf{c}, kR\mathbf{b}) = -k(\mathbf{a}, R\mathbf{b}).$$

Since under the rotation of both vectors through an angle of 90° their inner product does not vary and $R(R\mathbf{a}) = -\mathbf{a}$, it follows that

$$(\mathbf{b}, R\mathbf{a}) = (R\mathbf{b}, -\mathbf{a}) = -(\mathbf{a}, R\mathbf{b}).$$

7.5. Let O be the center of the sphere; M the center of mass of triangle ABC; $\mathbf{u} = \{SO\}$; let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be unit vectors directed along the edges of the trihedral angle. Then

$$3\{SM\} = \{SA\} + \{SB\} + \{SC\} = 2((\mathbf{u}, \mathbf{a})\mathbf{a} + (\mathbf{u}, \quad \mathbf{b})\mathbf{b} + (\mathbf{u}, \mathbf{c})\mathbf{c}).$$

The center O of the sphere belongs to the plane that passes through the midpoint of segment SN perpendicularly to it. Hence, $\mathbf{u} = \mathbf{e}_1 + \lambda\mathbf{e}_2 + \mu\mathbf{e}_3$, where $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are certain fixed vectors. Therefore,

$$3\{SM\} = 2(\varepsilon_1 + \lambda\varepsilon_2 + \mu\mathbf{e}_3), \quad \text{where } \varepsilon_i = (\mathbf{e}_i, \mathbf{a})\mathbf{a} + (\mathbf{e}_i, \mathbf{b})\mathbf{b} + (\mathbf{e}_i, \mathbf{c})\mathbf{c}.$$

7.6. Let $\mathbf{n}_1, \dots, \mathbf{n}_k$ be the unit outer normals to the faces; $M_1, \dots, M_k$ arbitrary points on these faces. The sum of the distances from an inner point X of the polyhedron to all the faces is equal to

$$\sum(\{XM_i\}, \mathbf{n}_i) = \sum(\{XO\}, \mathbf{n}_i) + \sum(\{OM_i\}, \mathbf{n}_i),$$

where O is a fixed inner point of the polyhedron. This sum does not depend on X only if

$$\sum(\{XO\}, \mathbf{n}_i) = 0, \text{ i.e., } \sum \mathbf{n}_i = \mathbf{0}.$$

7.7. Let O be the center of the circumscribed sphere of the orthocentric tetrahedron, H its orthocenter and M the center of mass.

Clearly, $\{OM\} = \frac{1}{4}(\{OA\} + \{OB\} + \{OC\} + \{OD\})$. Therefore, it suffices to verify that $\{OH\} = \frac{1}{2}(\{OA\} + \{OB\} + \{OC\} + \{OD\})$. Let us prove that if $\{OX\} = \frac{1}{2}(\{OA\} + \{OB\} + \{OC\} + \{OD\})$, then H is the orthocenter.

Let us prove, for instance, that $AX \perp CD$. Clearly,

$$\{AX\} = \{AO\} + \{OX\} = \frac{-\{OA\} + \{OB\} + \{OC\} + \{OD\}}{2} = \frac{\{AB\} + \{OC\} + \{OD\}}{2}.$$

Hence,

$$2(\{CD\} < \{AX\}) = (\{CD\},\ \{AB\} + \{OC\} + \{OD\}) = (\{CD\}, \{AB\}) + (-\{OC\} + \{OD\},\ \{OC\} + \{OD\}).$$

Both summands are equal to zero: the first one because $CD \perp AB$ and the second one because $OC = OD$. We similarly prove that $AX \perp BC$, i.e., line AX is perpendicular to face BCD.

For lines BX, CX and DX the proof is similar.

7.8. First solution. Let several rays with common origin O and forming pairwise obtuse angles be arranged in space. Let us introduce a coordinate system directing Ox-axis along the first ray and selecting for the coordinate plane Oxy the plane that contains the first two rays.

Each ray is determined by a vector $\mathbf{e}$ and instead of $\mathbf{e}$ we can as well take $\lambda\mathbf{e}$, where $\lambda > 0$. The first ray is given by vector $\mathbf{e}_1 = (1, 0, 0)$ and the k-th ray by vector

$\mathbf{e}_k = (x_k, y_k, z_k)$. For $k > 1$ the inner product of vectors $\mathbf{e}_1$ and $\mathbf{e}_k$ is negative; hence, $x_k < 0$. We may assume that $x_k = -1$.

Further, for $k > 2$ the inner product of vectors $\mathbf{e}_2$ and $\mathbf{e}_k$ is negative. Taking into account that $z_2 = 0$ thanks to the choice of the coordinate plane Oxy, we get $(\mathbf{e}_2, \mathbf{e}_k) = 1 + y_2y_k < 0$. Therefore, all the numbers y_k for $k > 2$ are of the same sign (opposite to the sign of y_2). Now, make use of the fact that

$$(\mathbf{e}_i, \mathbf{e}_j) = 1 + y_iy_j + z_iz_j < 0 \text{ for } i, j \geq 3 \text{ and } i \neq i.$$

Clearly, $y_iy_j > 0$; therefore, $z_iz_j < 0$. Since there are no three numbers of distinct signs, only two vectors distinct from the first two vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ can exist.

Second solution. First, let us prove that if

$$\lambda_1\mathbf{e}_1 + \cdots + \lambda_k\mathbf{e}_k = \lambda_{k+1}\mathbf{e}_{k+1} + \cdots + \lambda_n\mathbf{e}_n,$$

where all the numbers $\lambda_1, \dots, \lambda_n$ are positive and $1 \leq k < n$, then not all the angles between the vectors $\mathbf{e}_i$ are obtuse. Indeed, the squared length of vector $\lambda_1\mathbf{e}_1 + \cdots + \lambda_k\mathbf{e}_k$ is equal to

$$(\lambda_1\mathbf{e}_1 + \cdots + \lambda_k\mathbf{e}_k, \lambda_{k+1}\mathbf{e}_{k+1} + \cdots + \lambda_n\mathbf{e}_n)$$

and if all the angles between the vectors $\mathbf{e}_i$ are obtuse, then this inner product is the sum of negative numbers.

Now, suppose that there exist vectors $\mathbf{e}_1, \dots, \mathbf{e}_5$ in space all the angles between which are obtuse. Clearly, these vectors cannot be parallel to one plane; let for example, vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ be not parallel to one plane. Then

$$\mathbf{e}_4 = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2 + \lambda_3 + \mathbf{e}_3; \quad \mathbf{e}_5 = \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2 + \mu_3\mathbf{e}_3.$$

Let us subtract the second equality from the first one and rearrange the obtained equality so that in its right- and left-hand sides the vectors with positive coefficients would stand; then in the left-hand side $\mathbf{e}_4$ stands and in the right-hand side $\mathbf{e}_5$ stands. Contradiction.

7.9. Suppose that the angles between vectors $\mathbf{e}_1, \dots, \mathbf{e}_7$ are not acute ones. Let us direct Ox-axis along vectors $\mathbf{e}_1$. No plane perpendicular to $\mathbf{e}_1$ can have more than four vectors the angles between which are not acute; together with vector $-\mathbf{e}_1$ we get the total of only six vectors. Therefore, we can select a vector $\mathbf{e}_2$ and direct the Oy-axis so that $\mathbf{e}_2 = (x_2, y_2, 0)$, where $x_2 \neq 0$ (and, therefore, $x_2 < 0$) and $y_2 > 0$.

Let $\mathbf{e}_k = (x_k, y_k, z_k)$ for $k = 3, \dots, 7$. Then $x_k \leq 0$ and $x_kx_2 + y_ky_2 \leq 0$. Hence, $x_kx_2 \geq 0$ and, therefore, $y_ky_2 \leq 0$, i.e., $y_k \leq 0$. Since $(\mathbf{e}_s, \mathbf{e}_r) \leq 0$ for $3 \leq s$, $r \leq 7$ and $x_rx_s \geq 0$, $y_ry_s \geq 0$, it follows that $z_sz_r \leq 0$. But among the five numbers z_3, $\dots$, z_7 there are not more than two zero ones, hence, among the three remaining numbers there are necessarily two numbers of the same sign. Contradiction.

7.10. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_4$ be unit vectors perpendicular to faces and directed outwards; $\mathbf{n} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4$; s the indicated sum of the cosines. Since $(\mathbf{e}_i, \mathbf{e}_j) = -\cos\varphi_{ij}$, where φ_{ij} is the angle between the i-th and j-th faces then $|\mathbf{n}|^2 = 4 - 2s$. Now the inequality $s \leq 2$ is obvious. It remains to verify that $s > 0$, i.e., $|\mathbf{n}| \leq 2$.

There exist nonzero numbers α, β, γ and δ such that $\alpha\mathbf{e}_1 + \beta\mathbf{e}_2 + \gamma\mathbf{e}_3 + \delta\mathbf{e}_4 = \mathbf{0}$. Let, for definiteness, the absolute value of δ be the largest among these numbers.

Dividing the given equality by δ we may assume that $\delta = 1$. Then numbers α, β and γ are positive (cf. Problem 7.15 b)) and do not exceed 1. Since

$$\mathbf{n} = \mathbf{n} - \alpha\mathbf{e}_1 - \beta\mathbf{e}_2 - \gamma\mathbf{e}_3 - \mathbf{e}_4 = (1-\alpha)\mathbf{e}_1 + (1-\beta)\mathbf{e}_2 + (1-\gamma)\mathbf{e}_3,$$

it follows that

$$|\mathbf{n}| \leq 1 - \alpha + 1 - \beta + 1 - \gamma = 3 - (\alpha + \beta + \gamma).$$

It remains to notice that

$$1 = |\mathbf{e}_4| = |\alpha\mathbf{e}_1 + \beta\mathbf{e}_2 + \gamma\mathbf{e}_3| \leq \alpha + \beta + \gamma$$

and the equality cannot take place because the given vectors are not colinear.

7.11. Let vectors $\mathbf{a}_i$ and $\mathbf{b}_i$ be codirected with rays AA_i and Bb_i and are of unit length. By Problem 7.16 there exist positive numbers $x_1, \dots, x_n$ such that

$$x_1\mathbf{a}_1 + \dots + x_n\mathbf{a}_n = \mathbf{0}.$$

Consider vector

$$\mathbf{b} = x_1\mathbf{b}_1 + \dots + x_n\mathbf{b}_n.$$

Since $(\mathbf{b}_i, \mathbf{b}_j) \leq (\mathbf{a}_i, \mathbf{a}_j)$, it follows that by the hypothesis

$$|\mathbf{b}|^2 = \sum x_i^2 + 2\sum_{i<j} x_i x_j(\mathbf{b}_i, \mathbf{b}_j) \leq \sum x_i^2 + 2\sum_{i<j} x_i x_j(\mathbf{a}_i, \mathbf{a}_j) =$$
$$= |x_1\mathbf{a}_1 + \dots + x_n\mathbf{a}_n|^2 = 0.$$

If at least one of the inequalities $(\mathbf{b}_i, \mathbf{b}_j) \leq (\mathbf{a}_i, \mathbf{a}_j)$ is a strict one, we get a strict inequality $|\mathbf{b}|^2 < 0$ which is impossible.

7.12. Point X lies in plane ABC if and only if $\{AX\} = \lambda\{AB\} + \mu\{AC\}$, i.e.,

$$\{OX\} = \{OA\} + \{AX\} = \{OA\} + \lambda\{AB\} + \mu\{AC\} =$$
$$\{OA\} + \lambda(\{OB\} - \{OA\}) + \mu(\{OC\} - \{OA\}) =$$
$$(1 - \lambda - \mu)\{OA\} + \lambda\{OB\} + \mu\{OC\}.$$

Let point X belong to triangle ABC. Let us prove that, for example, $\lambda = S_{CXA} : S_{ABC}$. The equality $\{AX\} = \lambda\{AB\} + \mu\{AC\}$ means that the ratio of the heights dropped from points X and B to line AC is equal to λ and the ratio of these heights is equal to $S_{CXA} : S_{ABC}$.

7.13. Let $\mathbf{a} = \{AB\}$, $\mathbf{b} = \{AC\}$ and $\mathbf{c} = \{AD\}$. Further, let X be an arbitrary point and $\{AX\} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$. Point X belongs to plane KLM if

$$\{AX\} = p\{AK\} + q\{AL\} + r\{AM\} = \frac{p}{\alpha}\mathbf{a} + \frac{q}{\beta}\mathbf{b} + \frac{r}{\gamma}\mathbf{c}, \text{ where } p + q + r = 1$$

(cf. Problem 7.12), i.e.,

$$\lambda\alpha + \mu\beta + \nu\gamma = 1.$$

a) We have to select numbers λ, μ and ν so that for any α and β we would have had

$$\lambda\alpha + \mu\beta + \nu(\alpha + \beta + 1) = 1,$$

i.e.,

$$\lambda + \nu = 0, \quad \mu + \nu = 0 \quad \text{and} \quad \nu = 1.$$

b) Point X belongs to all the considered planes if

$$\lambda(\beta - 1) + \mu\beta + \nu(\beta + 1) = 1 \quad \text{for all} \quad \beta,$$

i.e.,

$$\lambda + \mu + \nu = 0 \quad \text{and } \nu - \lambda = 1.$$

Such points X fill in a straight line.

7.14. Let $\{OC\} = \lambda\{OA\} + \mu\{OB\}$. Then, since the regular pentagons are similar, $\{OC_1\} = \lambda\{OA_1\} + \mu\{OB_1\}$ and, therefore, $\{CC_1\} = \lambda\{AA_1\} + \mu\{BB_1\}$, i.e., line CC_1 is parallel to plane Π that contains $\{AA_1\}$ and $\{BB_1\}$.

We similarly prove that line DD_1 is parallel to plane Π.

7.15. a) In equality

$$\alpha\{OA\} + \beta\{OB\} + \gamma\{OC\} + \delta\{OD\} = \{0\},$$

let us transport all the summands with the negative coefficients to the right-hand side. If p, q and r are positive numbers, then the endpoint of vector $p\{OP\}+q\{OQ\}$ lies inside angle POQ and the endpoint of vector $p\{OP\} + q\{OQ\} + r\{OR\}$ lies inside the trihedral angle $OPQR$ with vertex O. It remains to notice that, for example, edge CD lies outside angle AOB and vertex D lies outside the trihedral angle $OABC$.

b) Since point O lies inside tetrahedron $A_1B_1C_1D_1$, we may make use of the solution of heading a).

7.16. Let the extension of ray OA_i beyond point O intersect the polyhedron at point M; let P be one of the vertices of the edge that contains point M; let QR be the side of this face that intersects with the extension of ray MP beyond point M. Then

$$\{OM\} = p\{OP\} + q\{OQ\} + r\{OR\}, \text{ where } p, q, r \geq 0.$$

Since vectors $\{OA_i\}$ and $\{OM\}$ have opposite directions,

$$\{OA_i\} + \alpha\{OP\} + \beta\{OQ\} + \gamma\{OR\} = \{0\},$$

where $\alpha, \beta, \gamma \geq 0$ and P, Q, R are some vertices of the polyhedron.

Write such equalities for all i from 1 to n and add them; we get the desired statement.

7.17. First solution. Any vector $\mathbf{u}$ can be represented in the form $\mathbf{u} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$; therefore, it suffices to carry out the proof for vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Since the center of a regular tetrahedron divides its median in the ratio of 1 : 3, we have

$$(\mathbf{a}, \mathbf{b}) = (\mathbf{a}, \mathbf{c}) = (\mathbf{a}, \mathbf{d}) = -\frac{1}{3}.$$

Taking into account that $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}=\mathbf{0}$ we get

$$(\mathbf{a},\mathbf{a})\mathbf{a}+(\mathbf{a},\mathbf{b})\mathbf{b}+(\mathbf{a},\mathbf{c})\mathbf{c}+(\mathbf{a},\mathbf{d})\mathbf{d}==\mathbf{a}-\frac{1}{3}(\mathbf{b}+\mathbf{c}+\mathbf{d})=\mathbf{a}+\frac{1}{3}\mathbf{a}=\frac{4}{3}\mathbf{a}.$$

For vectors $\mathbf{b}$ and $\mathbf{c}$ the proof is similar.

Second solution. Consider cube $ABCDA_1B_1C_1D_1$. Clearly, AB_1CD_1 is a regular tetrahedron. Introduce a rectangular coordinate system with the origin at the center of the cube and the axes parallel to the edges of the cube. Then

$$\sqrt{3}\mathbf{a}=(1,1,1),\quad \sqrt{3}\mathbf{b}=(-1,-1,1),\quad \sqrt{3}\mathbf{c}=(-1,1,-1)\ \text{ and } \sqrt{3}\mathbf{d}=(1,-1,-1).$$

Let $\mathbf{u}=(x,y,z)$. Easy but somewhat cumbersome calculations lead us now to the desired result.

7.18. Let us drop perpendiculars OB_i from point O to the faces of the tetrahedron. Let $\mathbf{a}_i$ be a unit vector directed as $\{OB_i\}$. Then $(\{OM\},\mathbf{a}_i)\mathbf{a}_i+\{MA_i\}=\{OB_i\}$. Since tetrahedron $B_1B_2B_3B_4$ is a regular one, the sum of vectors $\{OB_i\}$ is equal to zero. Therefore,

$$\sum\{MA_i\}=\sum(\{MO\},\mathbf{a}_i)\mathbf{a}_i=\frac{4\{MO\}}{3}$$

(see Problem 7.17).

7.19. First solution. Prove that the sum of the projections of all the given vectors to any line l is equal to zero. To this end consider the projection of the polyhedron to the plane perpendicular to line l. The projection of the polyhedron is covered by the projections of its faces in two coats since the faces can be divided into two types: "visible from above" and "visible from below" (we can disregard the faces whose projections are segments). Ascribe the "plus" sign to projections of the faces of one type and the "minus" sign to the projections of the other type we see that the sum of the signed areas of the projections of the faces is equal to zero.

Now, notice that the area of the projection of the face is equal to the length of the projection of the corresponding vector to line l (cf. Problem 2.13) and for faces of distinct types the projections of vectors have opposite directions. Therefore, the sum of projections of the vectors to line l is also equal to zero.

Second solution. Let X be a point inside the polyhedron, h_i the distance from X to the plane of the i-th face. Let us divide the polyhedron into pyramids with vertex X whose bases are the faces of the polyhedron. The volume V of the polyhedron is equal to the sum of volumes of these pyramids, i.e., $3V=\sum h_iS_i$, where S_i is the area of the i-th face.

Further, let $\mathbf{n}_i$ be the unit vector of the outer normal to the i-th face, M_i an arbitrary point of this face. Then $h_i=(\{XM_i\},\,\mathbf{n}_i)$ and, therefore,

$$3V=\sum h_iS_i=\sum(\{XM_i\},S_i\mathbf{n}_i)=\sum(\{XO\},S_i\mathbf{n}_i)+\sum(\{OM_i\},S_i\mathbf{n}_i)=$$
$$(\{OX\},\sum S_i\mathbf{n}_i)+3V.$$

Here O is a fixed point of the polyhedron. Therefore, $\sum S_i\mathbf{n}_i=\mathbf{0}$.

7.20. Consider parallelepiped $ABCDA_1B_1C_1D_1$. Let the diagonals of the faces with common edge BC lie on given lines and AC be one of these diagonals. Then

BC_1 is the other of such diagonals and B_1D the diagonal of the parallelepiped that lies on the third given line.

Let us introduce the rectangular coordinate system so that line AC coincides with the Ox-axis, line BC_1 is parallel to Oy-axis and passes through point $(0, 0, a)$, line B_1D is parallel to Oz-axis and passes through point $(a, a, 0)$. Then the coordinates of points A and C are $(x_1, 0, 0)$ and $(x_2, 0, 0)$; let the coordinates of points B and C_1 be $(0, y_1, a)$ and $(0, y_2, a)$, let those of points D and B_1 be (a, a, z_1) and (a, a, z_2), respectively. Since $\{AD\} = \{BC\} = \{B_1C_1\}$, it follows that

$$a - x_1 = x_2 = -a, \;\; a = -y_1 = y_2 - a \text{ and } z_1 = -a = a - z_2,$$

wherefrom

$$x_1 = 2a, \;\; x_2 = -a, \;\; y_1 = -a, \;\; y_2 = -2a, \;\; z_1 = -a \text{ and } z_2 = 2a.$$

Therefore, we have found the coordinates of vertices A, B, C, D, B_1 and C_1.

Simple calculations show that $AC = 3a$, $AB = a\sqrt{6}$ and $BC = a\sqrt{3}$, i.e., triangle ABC is a rectangular one and, therefore, the area of face $ABCD$ is equal to $AB \cdot BC = 3a^2\sqrt{2}$. The plane of face $ABCD$ is given by equation $y + z = 0$. The distance from point (x_0, y_0, z_0) to the plane $px + qy + rz = 0$ is equal, as we know (Problem 1.27), to

$$\frac{|px_0 + qy_0 + rz_0|}{\sqrt{p^2 + q^2 + r^2}}$$

and, therefore, the distance from point B_1 to face $ABCD$ is equal to $\frac{3}{\sqrt{2}}a$. Therefore, the volume of the parallelepiped is equal to $9a^3$.

7.21. Fix $a = |\mathbf{a}|$, $b = |\mathbf{b}|$ and $c = |\mathbf{c}|$. Let x, y, z be the cosines of the angles between vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{b}$ and $\mathbf{c}$, $\mathbf{c}$ and $\mathbf{a}$, respectively. Denote the difference between the left- and right-hand sides of the inequality to be proved by

$$f(x, y, z) = a + b + c + \sqrt{a^2 + b^2 + c^2 + 2(abx + bcy + acz)} - \sqrt{a^2 + b^2 + 2abx} - \\ - \sqrt{b^2 + c^2 + 2bcy} - \sqrt{c^2 + a^2 + 2acz}.$$

Numbers x, y and z are related by certain inequalities but it will be easier for us to prove that $f(x, y, z) \geq 0$ for all x, y, z whose absolute value does not exceed 1.

The function

$$\varphi(t) = \sqrt{p + t} - \sqrt{q + t} = \frac{p - q}{\sqrt{p + t} + \sqrt{q + t}}$$

is monotonous with respect to t. Therefore, for fixed y and z the function $f(x, y, z)$ attains the least value when $x = \pm 1$. Further, fix $x = \pm 1$ and z; in this case the function f attains the least value when $y = \pm 1$. Finally, fixing $x = \pm 1$ and $y = \pm 1$ we see that function f attains the least value when the numbers x, y, z are equal to ± 1. In this case vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are colinear and the inequality is easy to verify.

7.22. Statements a) and b) easily follow from the definitions.

c) **First solution.** Introduce a coordinate system $Oxyz$; direct the Ox-axis along vector $\mathbf{a}$. It is easy to verify that vector $(0, -az, ay)$ is the vector product of vectors $\mathbf{a} = (a, 0, 0)$ and $\mathbf{u} = (x, y, z)$. Indeed, vector $(0, -az, ay)$ is perpendicular to both vectors **a** and **u** and its length is equal to the product of the length of vectors **a** by the length of the height dropped to vector **a** from the endpoint of vector **u**. The compatibility of the orientations should be checked for distinct choices of signs of numbers y and z; we leave this to the reader.

Now, the required equality is easy to verify by expressing the coordinates of the vector products that enter it through the coordinates of vectors **b** and **c**.

Second solution. Consider prism $ABCA_1B_1C_1$, where $\{AB\} = \mathbf{b}$, $\{BC\} = \mathbf{c}$ and $\{AA_1\} = \mathbf{a}$. Since $\{AC\} = \mathbf{b} + \mathbf{c}$, the indicated equality means that the sum of the three vectors of the outer (or inner) normals to the lateral sides of the prism whose lengths are equal to the areas of the corresponding faces is equal to zero. Let $A'B'C'$ be the section of the prism by the plane perpendicular to a lateral edge. After the normal vectors are rotated through an angle of 90° in plane $A'B'C'$ they turn into vectors $d\{A'B'\}$, $d\{B'C'\}$ and $d\{C'A'\}$, where d is the length of the lateral edge of the prism. The sum of these vectors is, clearly, equal to zero.

7.23. Let $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$, where $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ are unit vectors directed along the coordinate axes. To solve the problem we can make use of the results of Problem 7.22 a)–c) but first observe that $[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3$, $[\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1$ and $[\mathbf{e}_3, \mathbf{e}_1] = \mathbf{e}_2$.

7.24. Both equalities can be proved by easy but somewhat cumbersome calculations with the help of the result of Problem 7.23.

7.25. a) By Problem 7.24 a)

$$[\mathbf{a}, [\mathbf{b}, \mathbf{c}]] = \mathbf{b}(\mathbf{c}c, \mathbf{a}) - \mathbf{c}(\mathbf{a}, \mathbf{b}),$$
$$[\mathbf{b}, [\mathbf{c}, \mathbf{a}]] = \mathbf{c}(\mathbf{a}, \mathbf{b}) - \mathbf{a}(\mathbf{b}, \mathbf{c});$$
$$[\mathbf{c}, [\mathbf{a}, \mathbf{b}]] = \mathbf{a}(\mathbf{b}, \mathbf{c}) - \mathbf{b}(\mathbf{a}, \mathbf{c}).$$

By adding up these equalities we get the desired statement.

b) Vectors $[\mathbf{b}, \mathbf{c}]$, $[\mathbf{c}, \mathbf{a}]$ and $[\mathbf{a}, \mathbf{b}]$ are perpendicular to plane ABC and codirected and their lengths are equal to $2S_{BOC}$, $2S_{COA}$ and $2S_{AOB}$, respectively. Hence, vectors $[\mathbf{a}, [\mathbf{b}, \mathbf{c}]]$, $[\mathbf{b}, [\mathbf{c}, \mathbf{a}]]$ and $[\mathbf{c}, [\mathbf{a}, \mathbf{b}]]$ being rotated through an angle of 90° in plane ABC turn into vectors $2\mathbf{a}S_{BOC}$, $2\mathbf{b}S_{COA}$ and $2\mathbf{c}S_{AOB}$, respectively.

7.26. Let **a**, **b** and **c** be vectors that determine three nonadjacent sides of the heptagon; $\mathbf{a}_1$, $\mathbf{b}_1$ and $\mathbf{c}_1$ the vectors of the opposite sides. Since $\mathbf{a}_1$ is perpendicular to **b** and **c**, it follows that $\mathbf{a}_1 = \lambda[\mathbf{b}, \mathbf{c}]$.

Therefore, the common perpendicular to vectors **a** and $\mathbf{a}_1$ is given by vector $\mathbf{n}_a = [\mathbf{a}, [\mathbf{b}, \mathbf{c}]]$. From the Jacobi identity it follows that $\mathbf{n}_a + \mathbf{n}_b + \mathbf{n}_c = \mathbf{0}$, i.e., these vectors are perpendicular to one line.

7.27. Let $\mathbf{a} = \{DA\}$, $\mathbf{b} = \{DB\}$ and $\mathbf{c} = \{DC\}$. The statement of the problem is equivalent to the equality

$$[\mathbf{a}, \mathbf{b}] + [\mathbf{b}, \mathbf{c}] + [\mathbf{c}, \mathbf{a}] + [\mathbf{b} - \mathbf{c}, \mathbf{a} - \mathbf{c}] = \mathbf{0}.$$

7.28. a) Let us prove that, for example, vector

$$[\mathbf{a}, \mathbf{b}] + [\mathbf{b}, \mathbf{c}] + [\mathbf{c}, \mathbf{a}]$$

lies in plane Π that passes through the bisector of face SAB perpendicularly to this face. Plane Π is perpendicular to vector $\mathbf{a} - \mathbf{b}$ and, therefore, it contains vector $[\mathbf{c}, \mathbf{a} - \mathbf{b}]$. Moreover, plane Π contains vector $[\mathbf{a}, \mathbf{b}]$; hence, it contains vector

$$[\mathbf{a}, \mathbf{b}] + [\mathbf{c}, \mathbf{a} - \mathbf{b}] = [\mathbf{a}, \mathbf{b}] + [\mathbf{b}, \mathbf{c}] + [\mathbf{c}, \mathbf{a}].$$

b) Let
$$\{OA\} = \{OA_1\}\sin\alpha_1 + \{OA_2\}\sin\alpha_2 + \{OA_3\}\sin\alpha_3.$$

Let us prove that, for example, plane OA_2A divides the angle between faces OA_2A_1 and OA_2A_3 in halves. To this end it suffices to verify that the perpendicular to plane OA_2A is the bisector of the angle between the perpendiculars to planes OA_2A_1 and OA_2A_3. The perpendiculars to these three planes are given by vectors

$$\{OA_2\}\times\{OA\} = \{OA_2\}\times\{OA_1\}\sin\alpha_1 + \{OA_2\}\times\{OA_3\}\sin\alpha_3,$$
$$\{OA_2\}\times\{OA_1\},\quad \{OA_2\}\times\{OA_3\},$$

respectively. As is easy to see, if $|\mathbf{a}| = |\mathbf{b}|$, then vector $\mathbf{a}+\mathbf{b}$ determines the bisector of the angle between vectors $\mathbf{a}$ and $\mathbf{b}$. Therefore, it remains to prove that the lengths of vectors $\{OA_2\}\times\{OA_1\}\sin\alpha_1$ and $\{OA_2\}\times\{OA_3\}\sin\alpha_3$ are equal. But

$$|\{OA_2\}\times\{OA_1\}| = \sin A_1OA_2 = \sin\alpha_3 \quad \text{and } |\{OA_2\}\times\{OA_3\}|\sin\alpha_1$$

which completes the proof. For planes OA_1A and OA_3A the proof is similar.

7.29. Let $\mathbf{a} = \{A_1B\}$, $\mathbf{b} = \{BC_1\}$ and $\mathbf{c} = \{C_1D\}$. Then the doubled areas of the faces of tetrahedron A_1BC_1D are equal to the lengths of vectors $[\mathbf{a},\mathbf{b}]$, $[\mathbf{b},\mathbf{c}]$, $[\mathbf{c},\mathbf{d}]$ and $[\mathbf{d},\mathbf{a}]$, where $\mathbf{d} = -(\mathbf{a}+\mathbf{b}+\mathbf{c})$ and the doubled areas of the faces of the parallelepiped are equal to the lengths of vectors $[\mathbf{a},\mathbf{c}]$, $[\mathbf{b},\mathbf{d}]$ and $[\mathbf{a}+\mathbf{b},\mathbf{b}+\mathbf{c}]$.

Let $\mathbf{x} = [\mathbf{a},\mathbf{b}]$, $\mathbf{y} = [\mathbf{b},\mathbf{c}]$ and $\mathbf{z} = [\mathbf{c},\mathbf{a}]$. Then four times the sums of the squares of areas of the faces of the tetrahedron and the parallelepiped are equal to

$$|\mathbf{x}|^2 + |\mathbf{y}|^2 + |\mathbf{y}-\mathbf{z}|^2 + |\mathbf{z}-\mathbf{x}|^2 \text{ and } |\mathbf{z}|^2 + |\mathbf{x}-\mathbf{y}|^2 + |\mathbf{x}+\mathbf{y}-\mathbf{z}|^2,$$

respectively. It is easy to verify that each of these sums is equal to

$$2(|\mathbf{x}|^2 + |\mathbf{y}|^2 + |\mathbf{z}|^2 - (\mathbf{y},\mathbf{z}) - (\mathbf{x},\mathbf{z})).$$

7.30. As is known, three vectors are complanar if and only if their mixed product is equal to zero. Making use of the formula from Problem 7.23 we see that the mixed product of the given vectors is equal to

$$(a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3.$$

7.31. Let M be the center of mass of the tetrahedron, A the midpoint of the edge through which plane Π passes, B the midpoint of the opposite edge, N' the point symmetric to N through point M. Since point M is the midpoint of segment AB (see Problem 14.3), it follows that $AN' \parallel BN$ and therfore point N' belongs to Π. Therefore, all the six planes pass through point N'.

7.32. a) Let A be the midpoint of edge a, B the midpoint of the opposite edge b. Further, let M be the center of mass of the tetrahedron, O the center of its circumscribed sphere, O' the point symmetric to O through M. Since point M is the midpoint of segment AB (Problem 14.3), it follows that $O'A \parallel OB$. But segment OB is perpendicular to edge b, hence, $O'A \perp b$ and, therefore, point O' belongs to the plane that passes through the midpoint of edge a perpendicularly to edge b. Therefore, all the 6 planes pass through point O'.

b) Let Monge's point O' lie in plane of face ABC. Let us draw plane Π parallel to this face through vertex D. Since the center O of the circumscribed sphere of the tetrahedron is symmetric to point O' through its center of mass M and point M divides the median of the tetrahedron drawn from vertex D in ratio $3:1$ (Problem 14.3), then point O is equidistant from planes Π and ABC. It remains to notice that if the center of the sphere is equidistant from the two parallel intersecting planes, then the projection of the circle of the section to the second intersecting plane coincides with the second circle of the section.

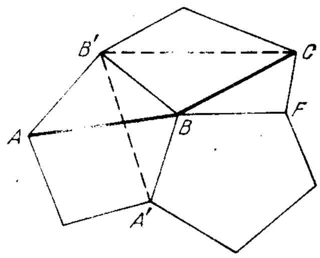

FIGURE 54 (SOL. 7.33)

7.33. Let us prove that $\angle ABC = 90°$ (Fig. 54). To this end let us consider the dashed segments $A'B'$ and $B'C$. Clearly, the symmetry through the plane that passes through the midpoint of segment BB' perpendicularly to it maps segment AB to $A'B'$ and BC to $B'C$. Therefore, it suffices to prove that $\angle A'B'C = 90°$. Moreover, $B'C \parallel BF$, i.e., we have to prove that $A'B' \perp BF$. The symmetry through the bisector plane of the dihedral angle formed by the pentagons with common edge BF sends point A' to B'. Therefore, segment $A'B'$ is perpendicular to this plane, in particular, $A'B' \perp BF$.

For the remaining angles between the considered segments the proof is carried out similarly.

7.34. First, suppose that both the given sphere and the sphere tangent to it lie in the same dihedral angle between the given planes. Then both spheres are symmetric through the bisector plane of this dihedral angle and, therefore, their tangent point lies in this plane. If the given sphere and the sphere tangent to it lie in distinct dihedral angles, then only one of the two tangent points of the given sphere with the given planes can be their common point. Therefore, the locus to be found is the union of the circle along which the bisector plane intersects the given sphere, and two tangent points of the given sphere with the given planes (it is easy to verify that all these points actually belong to the locus to be found).

7.35. Let α, β and γ be dihedral angles at edges OA, OB and OC, respectively. Consider point C' symmetric to C through O. In the trihedral angle $OABC'$ the dihedral angles at edges OA, OB and OC' are equal to $\pi-\alpha$, $\pi-\beta$ and γ. Plane OMC', where M is the midpoint of segment AB, divides the dihedral angle at edge OC' into two dihedral angles. Since planes OMP and OMQ, where P and Q are the midpoints of segments AC' and BC', respectively, are symmetry planes for trihedral angles $OAMC'$ and $OBMC'$, respectively, it follows that the indicated dihedral angles at edge OC' are equal to $\pi-\alpha$ and $\pi-\beta$. Therefore, $\gamma = (\pi-\alpha) + (\pi-\beta)$, as was required.

7.36. Let O be the projection of vertex S to the plane of the base of the pyramid. Since the vertices of the base of the pyramid are equidistant from point S, they are also equidistant from point O and, therefore, they lie on one circle with center O. Now, let us prove that $BC = AE$. Let M be the midpoint of side AB. Since $MO \perp AB$ and $SO \perp AB$, it follows that segment AB is perpendicular to plane SMO and, therefore, the symmetry through plane SMO sends segment SA to segment SB.

The dihedral angles at edges SA and SB are equal and, therefore, under this symmetry plane SAE turns into plane SBC. Since the circle on which the vertices of the base of the pyramid lie turns under this symmetry into itself, point E turns into point C.

We similarly prove that $BC = ED = AB = DC$.

7.37. Let Π be a symmetry plane of the figure consisting of three pair-wise nonparallel lines. Only two variants are possible:

1) Π is a symmetry plane for every given line;

2) one line is symmetric through Π and two other lines are symmetric to each other.

In the first case either one line is perpendicular to Π and the other two lines belong to Π or all the three lines belong to Π. Therefore, plane Π is determined by a pair of given lines. Hence, there are not more than 3 planes of symmetry of this type.

In the second case plane Π passes through the bisector of the angle between two of the given lines perpendicularly to the plane that contains these lines. For each pair of lines there exist exactly 2 such planes and, therefore, the number of planes of symmetry of this type is not more than 6.

Thus, there are not more than 9 planes of symmetry altogether. Moreover, the figure that consists of three pairwise perpendicular lines all passing through one point has precisely 9 planes of symmetry.

7.38. Let $\mathbf{a}'$ be the image of vector $\mathbf{a}$ under the considered symmetry; $\mathbf{u}$ the projection of vector $\mathbf{a}$ to the given line. Then $\mathbf{a}' + \mathbf{a} = 2\mathbf{u}$ and $\mathbf{u} = \mathbf{b}\frac{(\mathbf{a},\mathbf{b})}{(\mathbf{b},\mathbf{b})}$.

7.39. In space, introduce a coordinate system taking lines l_1 and l for Ox- and Oy-axes. The symmetry through line Ox sends point (x, y, z) to point $(x, -y, -z)$ and symmetry through line Oy sends the obtained point to point $(-x, -y, z)$.

7.40. Fix an axis of symmetry l. Let us prove that the remaining axes of symmetry can be divided into pairs. First, observe that symmetry through line l sends an axis of symmetry into an axis of symmetry. If axis of symmetry l' does not intersect l or intersects it not at a right angle, then the pair to l' is the axis symmetric to it through l. If l' intersects l at a right angle, then the pair to l' is the line perpendicular to l and l' and passing through their intersection point. Indeed, as follows from Problem 7.39, this line is an axis of symmetry.

7.41. Let M be the center of mass of the tetrahedron. The homothety with center M and coefficient $-\frac{1}{3}$ sends the vertices of the tetrahedron into the centers of mass of its faces and, therefore, the circumscribed sphere of the tetrahedron turns into a sphere of radius $\frac{R}{3}$ that intersects all the faces of the tetrahedron (or is tangent to it).

To prove that the radius of this sphere is not shorter than r, it suffices to draw planes parallel to the faces of the tetrahedron and tangent to the parts of the sphere situated outside the tetrahedron. Indeed, then this sphere would be inscribed in a tetrahedron similar to the initial one and not smaller than the initial one.

7.42. Let SAB be the initial face of pyramid $SABCD$, let SAD be its other face. Let us turn planes of these faces about lines AB and AD so that they coincide with the plane of the base (the rotation is performed through the lesser angle). Consider a coordinate system with the origin at point A and axes Ox and Oy directed along rays AB and AD, respectively. The first projection determines a transformation that sends point (x, y) to (x, ky), where $k = \cos\alpha$ with α being the angle between the base and a lateral face.

The second projection sends point (x, y) to (kx, y). Therefore, the composition of these transformation sends point (x, y) to (kx, ky).

7.43. Let A and B be the most distant from each other points of the polyhedron. Then the images of the polyhedron M under the homotheties with centers A and B and coefficient $\frac{1}{2}$ in each case determine the required disposition.

Indeed, these polyhedrons do not intersect since they are situated on distinct sides of the plane that passes through the midpoint of segment AB perpendicularly to it. Moreover, they lie inside M because M is a convex polyhedron.

7.44. Consider a convex polyhedron M and any three polyhedrons M_1, M_2 and M_3 homothetic to it with coefficient k. Let O_1, O_2 and O_3 be the centers of the corresponding homotheties. Clearly, if A is a point of polyhedron M most distant from the plane that contains points O_1, O_2 and O_3, then A does not belong to any of the polyhedrons M_1, M_2 and M_3. This follows from the fact that the homothety with coefficient k and center O that belongs to plane Π changes k times the greatest distance from the polyhedron to plane Π.

7.45. Let N be the center of mass of triangle ABC. The homothety with center N and coefficient $\frac{1}{4}$ sends point D to M. Let us prove that point M lies in plane Π that passes through the center O_1 of the circumscribed circle of triangle ABC perpendicularly to its median AK. Indeed, $OM \perp AK$ by the hypothesis and $OO_1 \perp AK$. Thus, point D lies in plane Π' obtained from plane Π under the homothety with center N and coefficient 4. Conversely, if point D lies in this plane, then $OM \perp AK$.

Further, let K and L be the midpoints of edges BC and AD. Then M is the midpoint of segment KL. Median OM of triangle KOL is a height only if $KO = OL$. Since $OA = OB$, the heights OK and OL of isosceles triangles BOC and AOD, respectively, are equal if and only if $BC = AD$, i.e., point D lies on the sphere of radius BC centered at A. The locus to be found is the intersection of this sphere with plane Π'.

7.46. We may assume that planes Π' and Π'' are not parallel since otherwise the statement is obvious. Let l be the intersection line of these planes, A_i^* the intersection point of l with plane $A_iA_i'A_i''$. Plane $A_iA_i'A_i''$ is perpendicular to l and, therefore, $l \perp A_i'A_i^*$ and $l \perp A_i''A_i^*$. Hence, if we rotate plane Π' about line l so that it would coincide with Π'', then lines $A_i'A_i''$ become perpendicular to l.

7.47. Consider the section with a plane perpendicular to line l. The desired statement now follows from the corresponding planimetric statement on the composition of two axial symmetries.

7.48. Let A be a point, B its image under the symmetry through point O, C the image of point B under the rotation through an angle of φ through line l and D the image of C under the symmetry through plane Π. Then D is the image of point A under the rotation through an angle of $180^\circ + \varphi$ through line l.

7.49. a) Let T be a transformation that sends point A to point B distinct from

A; let S be the symmetry through plane Π that passes through the midpoint of segment AB perpendicularly to it. Then

$$S \circ T(A) = S(B) = A,$$

i.e., A is a fixed point of transformation $S \circ T$. Moreover, if $T(X) = X$ for a point X, then $AX = T(A)T(X) = BX$. Therefore, point X belongs to Π; hence, $S(X) = X$. Thus, point A and all the fixed points of transformation T are also fixed points of transformation $S \circ T$.

In space, take 4 points not in one plane and consider their images under given transformation P. For $k \leq 4$ it is possible to select k transformations $S_1, \dots, S_k$ — symmetries through planes — such that the transformation $S_1 \circ \dots \circ S_k \circ P$ preserves the selected 4 points, i.e., this transformation preserves all the points in space. Therefore, $P = S_k \circ \dots \circ S_1$ and to prove this we can make use of the fact that if $S \circ F = G$, where S is the symmetry through a plane, then

$$S \circ C = S \circ S \circ F = F,$$

because $S \circ S$ is the identity transformation.

b) For a transformation that fixes O we can take O as one of the 4 points whose images determine this transformation. The rest of the proof is absolutely analogous to the solution of heading a).

7.50. a) By Problem 7.49 b) any movement of the first kind which has a fixed point is the composition of two symmetries through planes, i.e., is a rotation about the line along which these planes intersect (cf. Problem 7.47).

b) Let T be a given motion of the second kind, I the symmetry through a fixed point O of this transformation. Since we can represent I as the composition of three symmetries through three pairwise perpendicular planes passing through O, it follows that I is a second kind transformation. Therefore, $P = T \circ I$ is a first kind transformation, where O is a fixed point of this transformation. Therefore, P is a rotation about an axis l that passes through point O. Therefore, transformation $T = T \circ I \circ I = P \circ I$ is the composition of a rotation about a line l and the symmetry through a plane perpendicular to l (cf. Problem 7.48).

7.51. After the ball has rolled, any point A on its surface turns into a point $T(A)$, where T is a first kind movement with a fixed point, the center of the ball. By Problem 7.50 a), the movement T is a rotation about an axis l. Therefore, points X_1, X_2 and X_3 lie in the plane that passes through point X perpendicularly to l.

7.52. Let us relate with the given trihedral angle a rectangular coordinate system $Oxyz$. A ray of light that moves in the direction of vector (x, y, z) will move in the direction of vector $(x, y, -z)$ being reflected from plane Oxy. Therefore, after being reflected from all of its three faces it will move in the direction of vector $(-x, -y, -z)$.

7.53. Let B be the incidence point of the ray to the mirror; A the point on the ray distinct from B; K and L the projections of A to the mirror in the initial and rotated positions, respectively, A_1 and A_2 the points symmetric to A through these positions of the mirror.

The angle in question is equal to angle A_1BA_2. If $AB = a$, then $A_1B = A_2B = a$ and $AK = a \sin \alpha$. Since $\angle KAL = \beta$, then

$$A_1A_2 = 2KL = 2AK \sin \beta = 2a \sin \beta.$$

Therefore, if φ is the angle in question, then

$$\sin(\frac{\varphi}{2}) = \sin\alpha \sin\beta.$$

7.54. Let us introduce a coordinate system with the origin O in the vertex of the cone and axis Ox that passes through point A (Fig. 55).

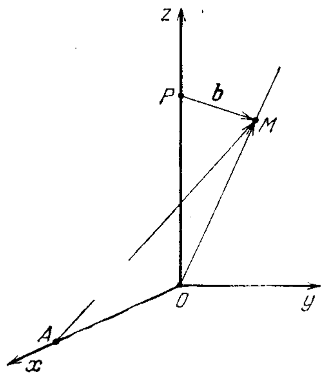

FIGURE 55 (SOL. 7.54)

Let $\{OM\} = (x, y, z)$, then $\{AM\} = (x - a, y, z)$, where $a = AO$. If α is the angle between axis Oz of the cone and the cone's generator, then $x^2 + y^2 = k^2z^2$, where $k = \tan\alpha$. Consider vector $\{PM\}$ perpendicular to the surface of the cone with the beginning point P on the axis of the cone. The coordinates of this vector are (x, y, t), where

$$0 = (\{OM\}, \{PM\}) = x^2 + y^2 + tz = k^2z^2 + tz, \text{ i.e., } \quad t = -k^2z.$$

The symmetry through line PM sends vector $\mathbf{a} = \{AM\}$ into vector $2\mathbf{b}\frac{(\mathbf{a},\mathbf{b})}{(\mathbf{b},\mathbf{b})} - \mathbf{a}$, where $\mathbf{b} = \{PM\}$ (cf. Problem 7.38). The third coordinate of this vector is equal to

$$-2k^2z\frac{x^2 - ax + y^2 - k^2z^2}{x^2 + y^2 + k^4z^2} - z = \frac{2ak^2xz}{(x^2 + y^2)(1 + k^2)} - z;$$

whereas it should be equal to zero. Therefore, the locus to be found is given by the equation

$$\frac{x^2 + y^2 - 2ak^2x}{1 + k^2} = 0.$$

It is the circle of radius $\frac{ak^2}{1+k^2} = a\sin^2 a$ that passes through the vertex of the cone.

CHAPTER 8. CONVEX POLYHEDRONS AND SPATIAL POLYGONS

§1. Miscellaneous problems

8.1. a) Areas of all the faces of a convex polyhedron are equal. Prove that the sum of distances from its inner point to the planes of the faces does not depend on the position of the plane.

b) The hights of the tetrahedron are equal to h_1, h_2, h_3 and h_4; let d_1, d_2, d_3 and d_4 be distances from an arbitrary inner point of the tetrahedron to the respective faces. Prove that

$$\sum \frac{d_i}{h_i} = 1.$$

8.2. a) Prove that a convex polyhedron cannot have exactly 7 edges.

b) Prove that a convex polyhedron can have any number of edges greater than 5 and distinct from 7.

8.3. A plane that intersects a circumscribed polyhedron divides it into two parts of volume V_1 and V_2; it divides its surface into two parts whose areas are S_1 and S_2. Prove that $V_1 : S_1 = V_2 : S_2$ if and only if the plane passes through the center of the inscribed sphere.

8.4. In a convex polyhedron, an even number of edges goes out from each vertex. Prove that any section of the polyhedron by a plane that does not contain its vertices is a polygon with an even number of sides.

8.5. Prove that if any vertex of a convex polyhedron is connected by edges with all the other vertices, then this polyhedron is a tetrahedron.

8.6. What is the greatest number of sides a projection of a convex polyhedron with n faces can have?

8.7. Each face of a convex polyhedron has a center of symmetry.

a) Prove that the polyhedron can be cut into parallelepipeds.

b) Prove that the polyhedron itself has the center of symmetry.

8.8. Prove that if all the faces of a convex polyhedron are parallelograms, then their number is the product of two consecutive positive integers.

§2. Criteria for impossibility to inscribe or circumscribe a polyhedron

8.9. Certain faces of a convex polyhedron are painted black, other faces are painted white so that no two black faces have a common edge. Prove that if the area of the black faces is greater than that of white ones, then no sphere can be inscribed into this polyhedron.

For a circumscribed polyhedron can the area of black faces be equal to that of white ones?

8.10. Certain faces of a convex polyhedron are painted black, the other ones white so that no two black faces have a common edge. Prove that if there are more black faces than whight ones, then it is impossible to inscribe this polyhedron into the sphere.

8.11. Some vertices of a convex polyhedron are painted black, the other ones are painted white so that at least one endpoint of each edge is white. Prove that if there are more black vertices than white ones, then this polyhedron cannot be inscribed in the sphere.

8.12. All the vertices of a cube are cut off by planes so that each plane cuts off a tetrahedron. Prove that the obtained polyhedron cannot be inscribed in a sphere.

8.13. Through all the edges of an octahedron planes are drawn so that a polyhedron with quadrilateral faces is obtained and to each edge of the octahedron one face corresponds. Prove that the obtained polyhedron cannot be inscribed in a sphere.

§3. Euler's formula

In this paragraph V is the number of vertices, E the number of edges, F the number of faces of a convex polyhedron.

8.14. Prove that $V - E + F = 2$. (*Euler's formula.*)

8.15. a) Prove that the sum of the angles of all the faces of a convex polyhedron is equal to the doubled sum of the angles of a plane polygon with the same number of vertices.

b) For every vertex of a convex polyhedron consider the difference between 2π and the sum of the plane angles at this vertex. Prove that the sum of all these differences is equal to 4π.

8.16. Let F_k be the number of k-gonal faces of an arbitrary polyhedron, V_k the number of its vertices at which k edges meet. Prove that

$$2E = 3V_3 + 4V_4 + 5V_5 + \dots = 3F_3 + 4F_4 + 5V_5 + \dots$$

8.17. a) Prove that in any convex polyhedron, there is either a triangular face or a trihedral angle.

b) Prove that for any convex polyhedron:

$$\#(\text{the triangular faces}) + \#(\text{the trihedral angles}) \geq 8.$$

8.18. Prove that in any convex polyhedron there exists a face that has not fewer than 6 sides.

8.19. Prove that for any convex polyhedron $3F \geq 6 + E$ and $3V \geq 6 + F$.

8.20. Given a convex polyhedron all whose faces have either 5, 6 or 7 sides and the polyhedral angles are all trihedral ones. Prove that the number of pentagonal faces is by 12 greater than the number of 7-gonal ones.

§4. Walks around polyhedrons

8.21. A planet is of the form of a convex polyhedron with towns at its vertices and roads between those towns along its edges. Two roads are closed for repairs. Prove that from any town one can reach any other town using the remaining roads.

8.22. On each edge of a convex polyhedron a direction is indicated; into any vertex at least one edge enters and at least one edge exits from it. Prove that there exist two faces such that one can go around them moving in accordance with the introduced orientation of the edges.

8.23. The system of roads that go along the edges of a convex polyhedron depicted on Fig. 56 connects all its vertices and divides it into two parts. Prove that this system of roads has no fewer than 4 deadends. (For the system of roads plotted on Fig. 56 vertices A, B, C and D correspond to the deadends.)

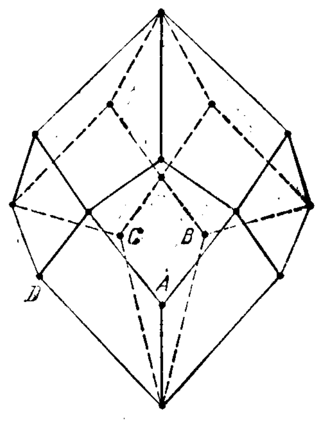

FIGURE 56 (8.22)

§5. Spatial polygons

8.24. A plane intersects the sides of a spatial polygon $A_1 \dots A_n$ (or their extensions) at points $B_1, \dots, B_n$, where point B_i lies on line A_iA_{i+1}. Prove that

$$\frac{A_1B_1}{A_2B_1} \cdot \frac{A_2B_2}{A_3B_2} \cdots \frac{A_nB_n}{A_1B_n} = 1$$

and the even number of points B_i lies on the sides of the polygon (not on their extensions).

8.25. Given four lines no three of which are parallel to one plane, prove that there exists a spatial quadrilateral whose sides are parallel to these lines and the ratio of the sides parallel to the corresponding lines for all such quadrilaterals is the same.

8.26. a) How many pairwise distinct spatial quadrilaterals with the same set of vectors of its sides are there?

b) Prove that the volumes of all the tetrahedrons determined by these spatial quadrilaterals are equal.

8.27. Givenoints A, B, C and D in space such that $AB = BC = CD$ and $\angle ABC = \angle BCD = \angle CDA = \alpha$. Find the angle between lines AC and BD.

8.28. Let $B_1, B_2, \dots, B_5$ be the midpoints of sides A_3A_4, A_4A_5, $\dots$, A_2A_3, respectively, of spatial pentagon $A_1 \dots A_5$; let also $\{A_iP_i\} = \left(1 + \frac{1}{\sqrt{5}}\right)\{A_iB_i\}$ and $\{A_iQ_i\} = \left(1 - \frac{1}{\sqrt{5}}\right)\{A_iB_i\}$. Prove that the points P_i as well as the points Q_i lie in one plane.

8.29. Prove that a pentagon all whose sides and angles are equal is a plane one.

* * *

8.30. In a spatial quadrilateral $ABCD$ the sums of the opposite sides are equal. Prove that there exists a sphere tangent to all its sides and diagonal AC.

8.31. A sphere is tangent to all the sides of the spatial quadrilateral. Prove that the tangent points lie in one plane.

8.32. On sides AB, BC, CD and DA of a spatial quadrilateral $ABCD$ (or on their extensions) points K, L, M and N, respectively, are taken so that $AN = AK$, $BK = BL$, $CL = CM$ and $DM = DN$. Prove that there exists a sphere tangent to lines AB, BC, CD and DA.

8.33. Let a, b, c and d be the lengths of sides AB, BC, CD and DA of spatial quadrilateral $ABCD$.

a) Prove that if none of the three relations

$$a + b = c + d, \quad a + c = b + d \quad \text{and } a + d = b + c$$

holds, then there exist exactly 8 distinct spheres tangent to lines AB, BC, CD and DA.

b) Prove that at least one of the indicated relations hold, then there exist infinitely many distinct spheres tangent to lines AB, BC, CD and DA.

Solutions

8.1. a) Let V be the volume of the polyhedron, S the area of its face, h_i the distance from point X inside the polyhedron to the i-th face. By dividing the polyhedron into pyramids with vertex X whose bases are its faces we get

$$V = \frac{Sh_1}{3} + \dots + \frac{Sh_n}{3}.$$

Therefore,

$$h_1 + \dots + h_n = \frac{3V}{S}.$$

b) Let V be the volume of the tetrahedron. Since $h_i = \frac{3V}{S_i}$, where S_i is the area of the i-th face, it follows that

$$\sum \frac{d_i}{h_i} = \frac{\sum d_i S_i}{3V}.$$

It remains to notice that $\frac{d_i S_i}{3} = V_i$, where V_i is the volume of the pyramid with vertex at the selected point of the tetrahedron, the i-th face is the base, and $\sum V_i = V$.

8.2. a) Suppose that the polyhedron has only triangular faces and their number is equal to F. Then the number of edges of the polyhedron is equal to $\frac{3F}{2}$, i.e., is divisible by 3. If the polyhedron has a face with more than 3 sides, then the polyhedron has not fewer than 8 edges.

b) Let $n \geq 3$. Then an n-gonal pyramid has $2n$ edges and the polyhedron obtained if we cut off a triangular pyramid in n-gonal pyramid with the plane that passes near one of the vertices of the base of the triangular pyramid has $2n + 3$ edges.

8.3. Suppose, for definiteness, that the center O of the inscribed sphere belongs to the part of the polyhedron with volume V_1. Consider the pyramid with vertex O whose base is the section of the polyhedron with the given plane. Let V be the volume of this pyramid. Then $V_1 - V = \frac{1}{3}rS_1$ and $V_2 + V = \frac{1}{3}rS_2$, where r is the radius of the inscribed sphere (cf. Problem 3.7). Therefore, $S_1 : S_2 = V_1 : V_2$ if and only if

$$(V_1 - V) : (V_2 + V) = S_1 : S_2 = V_1 : V_2$$

and, therefore $V = 0$, i.e., point O belongs to the intersecting plane.

8.4. There is a finite number of lines that connect vertices of the polyhedron and, therefore, we can jiggle the given plane a little so that in the process of jiggling it will not intersect any vertex and in its new position it will not be parallel to neither of the lines that connect the vertices of the polyhedron.

Let us move this plane parallel to itself until it stops intersecting the polyhedron. The number of vertices of the section will vary only when the plane will pass through the vertices of the polyhedron and each time it will pass one vertex only. If to one side of this plane there lies m edges that go out of the vertex and there are n edges on the other side, then the number of sides in the section when the vertex is passed changes by

$$n - m = (n + m) - 2m = 2k - 2m,$$

i.e., by an even number. Since after the plane leaves the polyhedron the number of the section's sides is equal to zero, the number of the sides of the initial section is an even one.

8.5. If any vertex of the polyhedron is connected by edges with any other vertices, then all the faces are triangular.

Consider two faces ABC and ABD with common edge AB. Suppose that the polyhedron is not a tetrahedron. Then it also has a vertex E distinct from the vertices of the considered faces. Since points C and D lie on different sides of plane ABE, triangle ABE is not a face of the given polyhedron.

If we cut the polyhedron along edges AB, BE and EA, then we divide the surface of the polyhedron into two parts (for a nonconvex polyhedron this would have been false) such that points C and D lie in distinct parts. Therefore, points C and D cannot be connected by an edge, since otherwise the cut would have intersected it but edges of a convex polyhedron cannot intersect along inner points.

8.6. Answer: $2n - 4$. First, let us prove that the projection of a convex polyhedron with n faces can have $2n - 4$ sides. Let us cut off regular tetrahedron $ABCD$ edge CD with a prismatic surface whose lateral edges are parallel to CD (Fig. 57). The projection of the obtained polyhedron with n faces to the plane parallel to lines AB and CD has $2n - 4$ sides.

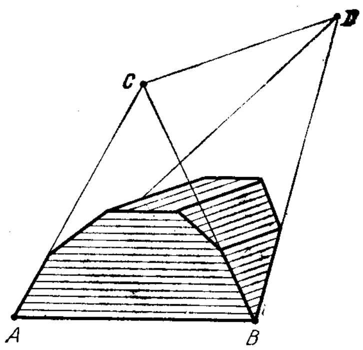

FIGURE 57 (SOL. 8.6)

Now, let us prove that the projection M of a convex polyhedron with n faces cannot have more than $2n - 4$ sides. The number of sides of the projection to the plane perpendicular to a face cannot be greater than the number of sides of all the other projections.

Indeed, such a projection sends the given face to a side of the polygon; if we slightly jiggle the plane of the projection, then this side will either be preserved or splits into several sides and the number of the remaining sides does not vary.

Therefore, we will consider the projections to planes not perpendicular to faces. In this case the edges whose projections belong to the boundary of the polygon M divide the polyhedron into two parts: the "upper" and the "lower". Let p_1 and p_2, q_1 and q_2, r_1 and r_2 be the numbers of vertices, edges and faces in the upper (subscript 1) and lower (subscript 2) parts, respectively (the vertices and edges on the boundary are ignored); m the number of vertices of M and m_1 (resp. m_2) the number of vertices of M from which at least one edge of the upper (resp. lower) part exits. Since from each vertex of M at least one edge of the upper or lower part exits, $m \leq m_1 + m_2$.

Now, let us estimate m_1. From each vertex of the upper part not less than 3 edges exit and, therefore, the number of the edges' endpoints for the upper part is not less than $3p_1 + m_1$.

On the other hand, the number of the endpoints of these edges is equal to $2q_1$; hence, $3p_1 + m_1 \leq 2q_1$. Now, let us prove that

$$p_1 - q_1 + r_1 = 1.$$

The projections of the edges of the upper part divide M into several polygons. The sum of the angles of these polygons is equal to $\pi(m-2) + 2\pi p_1$.

On the other hand, it is equal to $\sum_i \pi(q_{1i} - 2)$, where q_{1i} is the number of sides of the i-th polygon of the partition; the latter sum is equal to $\pi(m + 2q_1) - 2r_1$. By equating both expressions for the sum of the angles of the polygon we get the desired statement.

Since $q_1 = p_1 + r_1 - 1$ and $m_1 + 3p_1 \leq 2q_1$, it follows that $m_1 \leq 2r_1 - 2 - p_1 \leq 2r_1 - 2$. Similarly, $m_2 \leq 2r_2 - 2$. Therefore,

$$m \leq m_1 + m_2 \leq 2(r_1 + r_2) - 4 = 2n - 4.$$

8.7. a) Let us take an arbitrary face of the given polyhedron and its edge r_1. Since the face is centrally symmetric, it follows that it contains an edge r_2 equal and parallel to r_1. The face adjacent to edge r_2 also has an edge r_3 equal and parallel to r_1, etc. As a result we get a "belt" with faces determined by edge r_1. Show (this is not difficult) that it will necessarily close on edge r_1.

If we cut out this "belt" from the surface of the polyhedron then two "hats" remain: H_1 and H_2. Let us move hat H_1 inside the polyhedron by the vector determined by edge r_1 and cut the polyhedron along the surface $T(H_1)$ thus obtained. The parts of the polyhedron confined between H_1 and $T(H_1)$ can be divided into prisms and by dividing the bases of these prisms into parallelograms (as shown in Plane Problem 24.19) we get a partition into parallelepipeds.

The faces of the polyhedron confined between $T(H_1)$ and H_2 are centrally symmetric and the number of its edges is smaller than that of the initial polyhedron by the number of edges of the "belt" parallel to r_1. Therefore, after a finite number of such operations the polyhedron can be divided into parallelepipeds.

b) As in heading a) consider a "belt" and "hats" determined by an edge r of face F. The projection of the polyhedron to the plane perpendicular to edge r is a convex polygon whose sides are the projections of the faces that enter the "belt". The projections of faces from one hat determine a partition of this polygon into centrally symmetric polygons.

Therefore, this polygon is centrally symmetric itself (cf. Plane Problem 24.19), consequently, for edge E there exists an edge E' whose projection is parallel to the projection of E, i.e., these faces are parallel themselves; it is also clear that a convex polygon can only have one face parallel to E. Faces E and E' enter the same "belt"; therefore, E' also has an edge equal and parallel to edge r.

By performing similar arguments for all "belts" given by edges of face E we deduce that faces E and E' have corresponding equal and parallel edges. Since these faces are convex, they are equal. The midpoint of the segment that connects their centers of symmetry is their center of symmetry.

Thus, for any edge there exists a centrally symmetric face. It remains to demonstrate that all the centers of symmetry of pairs of faces coincide. It suffices to prove this for two faces with a common edge. By considering the "belt" determined by this edge we see that the faces parallel to them also have a common edge and both centers of symmetry of the pairs of faces coincide with the center of symmetry of the pair of common edges of these faces.

8.8. Let us make use of the solution of Problem 8.7. Each "belt" divides the surface of the polyhedron into two "hats". Since the polyhedron is centrally symmetric, both hats contain an equal number of faces. Therefore, another "belt" cannot lie entirely in one hat, i.e., any two belts intersect and the intersection constitutes precisely two faces (parallel to the edges that determine belts).

Let k be the number of distinct "belts". Then each "belt" intersects with $k-1$ other belts, i.e., it contains $2(k-1)$ faces. Since any face is a parallelogram, it enters exactly two belts. Therefore, the number of faces is equal to $\frac{2(k-1)k}{2} = (k-1)k$.

8.9. Let us prove that if no two black faces of the circumscribed polyhedron have a common edge, then the area of black faces does not exceed the area of white ones. In the proof we will make use of the fact that

if two faces of a polyhedron are tangent to the sphere at points O_1 and O_2 and AB is their common edge, then $\triangle ABO_1 = \triangle ABO_2$.

Let us divide the faces into triangles by connecting each tangent point of the polyhedron and the sphere with all the vertices of the corresponding face. From the preceding remark and the hypothesis it follows that to every black triangle we can associate a white triangle of the same area. Therefore, the sum of the areas of black triangles is not less than the sum of the areas of the white triangles.

The circumscribed polyhedron — a regular octahedron — can be painted so that the area of the black faces is equal to the area of the white ones and no two black faces have a common edge.

8.10. Let us prove that if a sphere is inscribed into the polyhedron and no two black faces have a common edge, then there are not more black faces than there are white ones. In the proof we will make use of the fact that

if O_1 and O_2 are tangent points with the sphere of faces with common edge AB, then $\triangle ABO_1 = \triangle ABO_2$ and, therefore, $\angle AO_1B = \angle AO_2B$.

For all the faces consider all the angles that subtend the edges of a face, the angles with vertices at the tangent points of the sphere with this face. From the preceding remark and the hypothesis it follows that to each such angle of a black face we can associate an equal angle of a white face. Therefore, the sum of black angles does not exceed the sum of white angles.

On the other hand, the sum of such angles for one face is equal to 2π. Hence, the sum of black angles is equal to $2\pi n$, where n is the number of black faces, and the number of white angles is equal to $2\pi m$, where m is the number of white faces. Thus, $n \leq m$.

8.11. Let us prove that if the polyhedron is inscribed in a sphere and no two black vertices are connected by an edge, then the number of black vertices does not exceed the number of white ones.

Let the planes tangent to the sphere centered at O at points P and Q intersect along line AB. Then any two planes passing through segment PQ cut on plane ABP the same angle as on plane ABQ. Indeed, these angles are symmetric through plane ABO.

Now, for each vertex of our polyhedron consider the angles that dihedral angles between the faces at this vertex cut on the tangent plane. From the preceding remark and the hypothesis it follows that to every angle at a black vertex we can associate an equal angle at a white vertex. Therefore, the sum of black angles does not exceed the sum of white ones.

On the other hand, the sum of such angles for one vertex is equal to $\pi(n-2)$, where n is the number of faces of the polyhedral angle at this vertex (to prove this it is convenient to consider the section of the polyhedral angle by a plane parallel to the tangent plane). We also see that if instead of these angles we consider the angles complementing them to 180° (i.e., the exterior angles of the polyhedron of the section), then their sum for any vertex will be equal to 2π. As earlier, the sum of such black angles does not exceed the sum of such white angles.

On the other hand, the sum of black angles is equal to $2\pi n$, where n is the number of black vertices, and the sum of white angles is equal to $2\pi m$, where m is the number of white vertices. Therefore, $2\pi n \leq 2\pi m$, i.e., $n \leq m$.

8.12. Let us paint the faces of the initial cube white and the remaining faces of the obtained polyhedron black. There are 6 white faces and 8 black faces and no two black faces have a common edge. Therefore, it is impossible to inscribe a sphere in this polyhedron (cf. Problem 8.10).

8.13. Let us paint 6 vertices of the initial octahedron white and 8 new vertices black. Then one endpoint of each edge of the obtained polyhedron is white and the other one is black. Therefore, it is impossible to inscribe this polyhedron into a sphere (cf. Problem 8.11).

8.14. First solution. Let M be the projection of the polyhedron to the plane not perpendicular to any of its faces; this projection maps all the faces to polygons. The edges that go into sides of the boundary of M divide the polyhedron into two parts. Let us consider the projection of one of these parts (Fig. 58). Let $n_1, \dots,$ n_k be the numbers of edges of the faces of this part, V_1 the number of the inner vertices of this part, V' the number of vertices on the boundary of M.

The sum of the angles of the polygons into which the polygon M is divided is, on the one hand, equal to $\sum \pi(n_i - 2)$ and, on the other hand, to $\pi(V'-2) + 2\pi V_1$. Therefore,

$$\sum n_i - 2k = V' - 2 + 2V_1,$$

where k is the number of faces in the first part. Writing down a similar equality for the second part of the polyhedron and taking their sum we get the desired statement.

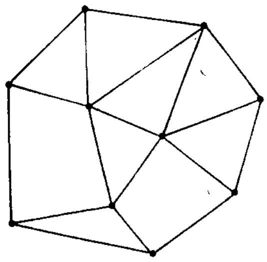

FIGURE 58 (SOL. 8.14)

Second solution. Let us consider the unit sphere whose center O lies inside the polyhedron. The angles of the form AOB, where AB is an edge of the polyhedron, divide the surface of the sphere into spherical triangles.

Let n_i be the number of sides of the i-th spherical polygon, σ_i the sum of its angles, S_i its area. By Problem 4.44 $S_i = \sigma_i - \pi(n_i - 2)$. Summing all these equalities for $i = 1, \dots, F$ we get

$$4\pi = 2\pi V - 2\pi E + 2\pi F.$$

8.15. Let Σ be the sum of all the faces of a convex polyhedron. In heading a) we have to prove that $\Sigma = 2(V-2)\pi$ and in heading b) we have to prove that $2V\pi - \Sigma = 4\pi$. Therefore, the headings are equivalent.

If a face has k edges, then the sum of its angles is equal to $(k-2)\pi$. When we sum over all the faces every edge is counted twice because it belongs to precisely two faces. Therefore, $\Sigma = (2E - 2F)\pi$. Hence,

$$2V\pi - \Sigma = 2\pi(V - E + F) = 4\pi.$$

8.16. To every edge we can associate two vertices that it connects. The vertex in which k edges meet is encountered k times. Therefore,

$$2E = 3V_3 + 4V_4 + 5V_5 + \dots$$

On the other hand, to every edge we can associate two faces adjacent to it, hence, a k-gonal face is encountered k times. Therefore,

$$2E = 3F_3 + 4F_4 + 5F_5 + \dots$$

8.17. a) Suppose that a convex polyhedron has neither triangular faces nor trihedral angles. Then $V_3 = F_3 = 0$; therefore, $2E = 4F_4 + 5F_5 + \dots \geq 4F$ and $2E = 4V_4 + 5V_5 + \dots \geq 4V$ (see Problem 8.16). Thus, $4V - 4E + 4F \leq 0$. On the other hand, $V - E + F = 2$. Contradiction.

b) By Euler's formula $4V + 4F = 4E + 8$. Let us substitute into this formula the following expressions for its constituents:

$$4V = 4V_3 + 4V_4 + 4V_5 + \dots, \quad 4F = 4F_3 + 4F_4 + 4F_5 \dots$$
$$4E = 2E + 2E = 3V_3 + 4V_4 + 5V_5 + \dots + 3F_3 + 4F_4 + 5F_5 + \dots$$

After simplification we get

$$V_3 + F_3 = 8 + V_5 + 2V_6 + 3V_7 + \cdots + F_5 + 2F_6 + 3F_7 + \cdots \geq 8.$$

8.18. Suppose that any face of a convex polyhedron has at least 6 sides. Then $F_3 = F_4 = F_5 = 0$ and, therefore, $2P = 6F_6 + 7F_7 + \cdots \geq 6F$ (cf. Problem 8.16), i.e., $E \geq 3F$. Moreover, for any polyhedron we have

$$2E = 3V_3 + 4V_4 + \cdots \geq 3V.$$

By adding the inequalities $E \geq 3F$ and $2E \geq 3V$ we get $E \geq F + V$. On the other hand, $E = F + V - 2$. Contradiction.

Remark. We can similarly prove that in any convex polyhedron there exists a vertex at which at least 6 edges meet.

8.19. For any polyhedron we have

$$2E = 3V_3 + 4V_4 + 5V_5 + \cdots \geq 3V.$$

On the other hand, $V = E - F + 2$. Therefore, , $2E \geq 3(E - F + 2)$, i.e., $3F \geq 6 + E$. The inequality $3V \geq 6 + E$ is similarly proved.

8.20. Let a, b and c be the total number of faces with 5, 6 and 7 sides, respectively. Then

$$E = \frac{5a + 6b + 7c}{2}, \quad F = a + b + c$$

and since by the hypothesis at every vertex 3 edges meet, $V = \frac{5a+6b+7c}{3}$. Multiplying these expressions by 6 and inserting them into the formula $6(V + F - E) = 12$ we get the desired statement.

8.21. Let A and B be the given towns. First, let us prove that one could ride from A to B along the roads before the two roads were closed for repairs. To this end let us consider the projection of the polyhedron to a line not perpendicular either of the polyhedron's edges (such a projection does not send distinct vertices of the polyhedron into one point).

Let A' and B' be projections of points A and B, respectivly, and M' and N' be the extremal points of the projection of the polyhedra; let M and N be vertices whose projections are M' and N', respectively. If we go from vertex A so that the movement in the projection is performed in the direction from M' to N', then in the end we will necessarily get to vertex N. Similarly, from vertex B we can reach N. Thus, we can get from A to B (via N).

If the obtained road from A to B passes along the road to be closed, then there are two more roundabout ways along the faces for which this edge is a common one. The second closed road cannot simultaneously go over both of these roundabouts.

8.22. Let us go out of a vertex of the polyhedron and continue walking along the edges in the direction indicated on them until we get a vertex where we have already been. The road from the first passage through this vertex to the second one forms a "loop" that divides a polyhedron into two parts. Let us consider one of them. On it, let us find a face with the desired property.

It is possible to circumvent the boundary of each of the two parts by moving in accordance with the introduced orientation. If the considered figure is a face itself, then everything is proved.

Therefore, let us assume that it is not a face, i.e., its boundary has a vertex from (resp. at) which an edge that does not belong to the boundary of the figure exits (resp. enters). Let us go along this edge and continue to go further along the edges in the indicated directions (resp. in the directions opposite to the indicated ones) until we again reach the boundary or get a loop. This pass divides the figure into two parts; the boundary of one of them can be circumvent in accordance with the orientation (Fig. 59). With this part perform the same operation, etc.

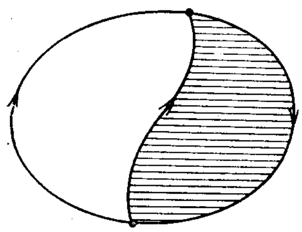

FIGURE 59 (SOL. 8.22)

After several such operations there remains one face that possesses the desired property. For the other of the parts obtained at the very first stage we can similarly find another of the required faces.

8.23. Let us paint the vertices of the polyhedron two colours as indicated on Fig. 60. Then any edge connects two vertices of distinct colours. For the given system of roads call the number of roads that pass through a vertex of the polyhedron *the degree of the vertex.*

If the system of roads has no vertices of degree greater than 2, then the difference between the number of black and white vertices does not exceed 1.

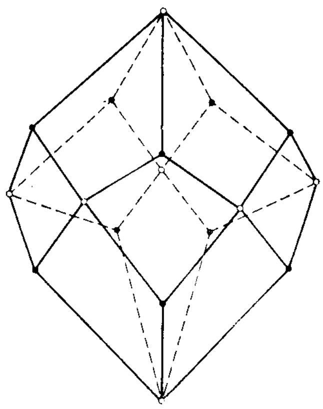

FIGURE 60 (SOL. 8.23)

If there is at least one vertex of degree 3 and the degrees of the other vertices do not exceed 2, then the difference between the number of black and white vertices does not exceed 2. In our case the difference between the number of black and white vertices is equal to $10 - 7 = 3$. Hence, there exists a vertex of degree not less than 4 or 2 vertices of degree 3. In either case the number of deadends is not fewer than 4.

8.24. Let us consider the projection to a line perpendicular to the given plane. The projections of all the points B_i is one point, B, and the projections of points $A_1, \dots, A_n$ are $C_1, \dots, C_n$, respectively. Since the ratios of the segments that lie on one line are preserved under a projection,

$$\frac{A_1B_1}{A_2B_1} \cdot \frac{A_2B_2}{A_3B_2} \dots \frac{A_nB_n}{A_1B_n} = \frac{C_1B}{C_2B} \cdot \frac{C_2B}{C_3B} \dots \frac{C_nB}{C_1B} = 1.$$

The given plane divides the space into two parts. By going from vertex A_i to A_{i+1} we pass from one part of the space to another one only if point B_i lies on side A_iA_{i+1}. Since by going over the polyhedron we return to the initial part of the space, the number of points B_i that lie on the sides of the polyhedron is an even one.

8.25. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be vectors parallel to the given lines. Since any three vectors in space not in one plane form a basis, there exist nonzero numbers α, β and γ such that $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} + \mathbf{d} = \mathbf{0}$. Vectors $\alpha\mathbf{a}$, $\beta\mathbf{b}$, $\gamma\mathbf{c}$ and $\mathbf{d}$ are sides of the quadrilateral to be found.

Now, let $\alpha_1\mathbf{a}$, $\beta_1\mathbf{b}$, $\gamma_1\mathbf{c}$ and $\mathbf{d}$ be vectors of the sides of another such quadrilateral. Then

$$\alpha_1\mathbf{a} + \beta_1\mathbf{b} + \gamma_1\mathbf{c} + \mathbf{d} = \mathbf{0} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} + \mathbf{d},$$

i.e.,

$$(\alpha_1 - \alpha)\mathbf{a} + (\beta_1 - \beta)\mathbf{b} + (\gamma_1 - \gamma)\mathbf{c} = \mathbf{0}.$$

Since vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ do not lie in one plane, it follows that $\alpha = \alpha_1$, $\beta = \beta_1$ and $\gamma = \gamma_1$.

8.26. a) Fix one of the vectors of sides. It can be followed by any of the three of remaining vectors which can be followed by any of the remaining vectors. Therefore, the total number of distinct quadrilaterals is equal to 6.

b) Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be given vectors of sides. Let us consider a parallelepiped determined by vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ (Fig. 61); vector $\mathbf{d}$ serves as its diagonal. An easy case-by-case checking demonstrates that all the 6 distinct quadrilaterals are contained among the quadrilaterals whose sides are the faces of this parallelepiped and its diagonal is d (when performing this case-by-case checking it is convenient to fix vector $\mathbf{d}$). The volume of the corresponding tetrahedron constitutes $\frac{1}{6}$ of the volume of the parallelepiped.

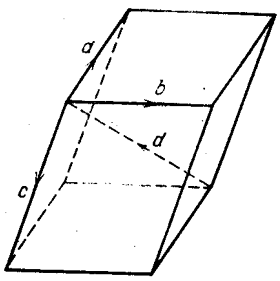

FIGURE 61 (SOL. 8.26)

8.27. In triangles ABC and CDA, sides AB and CD and angles B and D are equal and side AC is the common one. If $\triangle ABC = \triangle CDA$, then $AC \perp BD$.

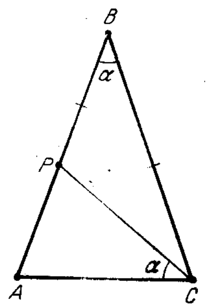

FIGURE 62 (SOL. 8.27)

Now, consider the case when these triangles are not equal. On ray BA, take point P such that $\triangle CBP = \triangle CDA$, i.e., $CP = CA$ (Fig. 62). Point P might not coincide with point A only if $\angle ABC < \angle APC = \angle BAC$, i.e., $\alpha < 60^\circ$. In this case

$$\angle ACD = \angle PCB = \left(90^\circ - \frac{\alpha}{2}\right) - \alpha = 90^\circ - \frac{3\alpha}{2}.$$

Therefore,

$$\angle ACD + \angle DCB = \left(90^\circ - \frac{3\alpha}{2}\right) + \alpha = 90^\circ - \frac{\alpha}{2} = \angle ACB.$$

Hence, points A, B, C and D lie in one plane and point D lies inside angle ACB. Since $\triangle ABC = \triangle DCB$ and these triangles are isosceles ones, the angle between lines AC and BD is equal to α.

Thus, if $\alpha \geq 60^\circ$, then $AC \perp BD$ and if $\alpha < 60^\circ$, then either $AC \perp BD$ or the angle between lines AC and BD is equal to α.

8.28. Let $\{A_iX_i\} = \lambda\{A_iB_i\}$. It suffices to verify that for $\lambda = 1 \pm \frac{1}{\sqrt{5}}$ the sides of the pentagon $X_1 \dots X_5$ are parallel to the opposite diagonals. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{e}$ be the vectors of the sides $\{A_1A_2\}$, $\{A_2A_3\}$, $\dots$, $\{A_5A_1\}$. Then

$$\begin{aligned}
\{A_1X_1\} &= \lambda\left(\mathbf{a}+\mathbf{b}+\tfrac{\mathbf{c}}{2}\right),\\
\{A_1X_2\} &= \mathbf{a}+\lambda\left(\mathbf{b}+\mathbf{c}+\tfrac{\mathbf{d}}{2}\right),\\
\{A_1X_3\} &= \mathbf{a}+\mathbf{b}+\lambda\left(\mathbf{c}+\mathbf{d}+\tfrac{\mathbf{e}}{2}\right),\\
\{A_1X_4\} &= \mathbf{a}+\mathbf{b}+\mathbf{c}+\lambda\left(\mathbf{d}+\mathbf{e}+\tfrac{\mathbf{a}}{2}\right);\\
\{A_1X_5\} &= \mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}+\lambda\left(\mathbf{e}+\mathbf{a}+\tfrac{\mathbf{b}}{2}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\{X_1X_3\} &= \{A_1X_3\} - \{A_1X_1\} = (1-\lambda)\mathbf{a} + (1-\lambda)\mathbf{b} + \lambda\mathbf{d} + \tfrac{\lambda}{2}(\mathbf{c}+\mathbf{e}) = \\
&\qquad \left(1-\tfrac{3\lambda}{2}\right)\mathbf{a} + \left(1-\tfrac{3\lambda}{2}\right)\mathbf{b} + \tfrac{\lambda}{2}\mathbf{d},\\
\{X_4X_5\} &= \{A_1X_5\} - \{A_1X_4\} = \tfrac{\lambda}{2}\mathbf{a} + \tfrac{\lambda}{2}\mathbf{b} + (1-\lambda)\mathbf{d}.
\end{aligned}$$

Thus, $X_1X_3 \parallel X_4X_5$ if and only if

$$\frac{2-3\lambda}{\lambda} = \frac{\lambda}{2-2\lambda},$$

i.e.,

$$5\lambda^2 - 10\lambda + 4 = 0.$$

The roots of this equation are $1 \pm \frac{1}{\sqrt{5}}$.

8.29. First solution. Suppose that the given pentagon $A_1 \dots A_5$ is not a plane one. The convex hull of its vertices is either a quadrilateral pyramid or consists of two tetrahedrons with the common face. In both cases we may assume that vertices A_1 and A_4 lie on one side of plane $A_2A_3A_5$ (see Fig. 63).

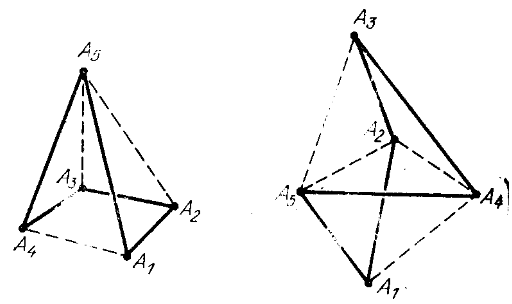

FIGURE 63 (SOL. 8.29)

It follows from the condition of the problem hat the diagonals of the given pentagon are equal because the tetrahedrons $A_4A_2A_3A_5$ and $A_1A_3A_2A_5$ are equal. Since points A_1 and A_4 lie on one side of face $A_2A_3A_5$ — an isosceles triangle — it follows that A_1 and A_4 are symmetric through the plane that passes through the midpoint of segment A_2A_3 perpendicularly to it. Therefore, points A_1, A_2, A_3 and A_4 lie in one plane.

Now, by considering equal (plane) tetrahedrons $A_1A_2A_3A_4$ and $A_1A_5A_4A_3$ we come to a contradiction.

Second solution. Tetrahedrons $A_1A_2A_3A_4$ and $A_2A_1A_5A_4$ are equal because their corresponding edges are equal. These tetrahedrons are symmetric either through the plane that passes through the midpoint of segment A_1A_2 perpendicularly to it or through line A_4M, where M is the midpoint of segment A_1A_2.

In the first case diagonal A_3A_5 is parallel to A_1A_2 and, therefore, 4 vertices of the pentagon lie in one plane. If there are two diagonals with such a property, then the pentagon is a plane one.

If there are 4 diagonals with the second property, then two of them go out of one vertex, say, A_3. Let M and K be the midpoints of sides A_1A_2 and A_4A_5, let L and N be the midpoints of diagonals A_1A_3 and A_3A_5, respectively. Since segment A_3A_5 is symmetric through line A_4M, its midpoint N lies on this line. Therefore, points A_4, M, N, A_3 and A_5 lie in one plane; the midpoint K of segment A_4A_5 lies in the same plane.

Similarly, points A_2, K, L, A_3, A_1 and M lie in one plane. Therefore, all the vertices of the pentagon lie in plane A_3KM.

8.30. Let the inscribed circles S_1 and S_2 of triangles ABC and ADC be tangent to side AC at points P_1 and P_2, respectively. Then

$$AP_1 = \frac{AB + AC - BC}{2} \text{ and } AP_2 = \frac{AD + AC - CD}{2}.$$

Since $AB - BC = AD - CD$ by the hypothesis, then $AP = AP_2$, i.e., points P_1 and P_2 coincide. Therefore, circles S_1 and S_2 lie on one sphere (cf. Problem 4.12).

8.31. Let the sphere be tangent to sides AB, BC, CD and DA of the spatial quadrilateral $ABCD$ at points K, L, M and N, respectively. Then $AN = AK$, $BK = BL$, $CL = CM$ and $DM = DN$. Therefore,

$$\frac{AK}{BK} \cdot \frac{BL}{CL} \cdot \frac{CM}{DM} \cdot \frac{DN}{AN} = 1.$$

Now, consider point N' at which plane KLM intersects with line DA. By making use of the result of Problem 8.24 we get $DN : AN = DN' : AN'$ and point N' lies on segment AD. It follows that $N = N'$, i.e., point N lies in plane KLM.

8.32. Since $AN = AK$, in plane DAB there is a circle S_1 tangent to lines AD and AB at points N and K, respectively. Similarly, in plane ABC there is a circle S_2 tangent to lines AB and BC at points K and L, respectively.

Let us prove that the sphere on which circles S_1 and S_2 lie is the desired one. This sphere is tangent to lines AD, AB and BC at points N, K and L, respectively (in particular, points B, C and D lie outside this sphere). It remains to verify that this sphere is tangent to line CD at point M.

Let S_3 be the section of the given sphere by plane BCD, let DN' be the tangent to S_3. Since $DC = \pm DM \pm MC$, $DM = DN = DN'$ and $MC = CL$, then the length of segment DC is equal to the sum or the difference of the lengths of the tangents drawn to S_3 from points C and D. This means that line CD is tangent to S_3. Indeed, let $a = d^2 - R^2$, where d is the distance from the center of S_3 to line CD and R be the radius of S_3; let P be the base of the perpendicular dropped from the center of S_3 to line CD; let $x = CD$ and $y = DP$. Then the lengths of the tangents CL and DN' are equal to $\sqrt{x^2 + a}$ and $\sqrt{y^2 + a}$. Let

$$|\sqrt{x^2 + a} \pm \sqrt{y^2 + a}| = |x \pm y| \neq 0.$$

Let us prove then that $a = 0$. By squaring both sides we get

$$\sqrt{(x^2 + a)(y^2 + a)} = \pm xy \pm a.$$

By squaring once again we get

$$a(x^2 + y^2) = \pm 2axy.$$

If $a \neq 0$, then $(x \pm y)^2 = 0$, i.e., $x = \pm y$. The equality $2|\sqrt{x^2 + a}| = 2|x|$ holds only if $a = 0$.

8.33. a) On lines AB, BC, CD and DA, introduce coordinates taking points A, B, C and D, respectively, for the origins and directions of rays AB, BC, CD and DA for the positive directions. In accordance with the result of Problem 8.32 let us search for lines AB, BC, CD and DA for points K, L, M and N, respectively, such that $AN = AK$, $BK = BL$, $CL = CM$ and $DM = DN$, i.e.,

$$\begin{array}{llllllll} \{AK\} & = \mathbf{x}, & \{AN\} & = \alpha\mathbf{x}, & \{BC\} & = \mathbf{y}, & \{BK\} & = \beta\mathbf{y}, \\ \{CM\} & = \mathbf{z}, & \{CL\} & = \gamma\mathbf{z}, & \{DN\} & = \mathbf{u}, & \{DM\} & = \delta\mathbf{u}, \end{array}$$

where $\alpha, \beta, \gamma, \delta = \pm 1$. Since $\{AB\} = \{AK\} + \{KB\}$, it follows that $a = x + \beta y$. Similarly,

$$b = y - \gamma z, \quad c = z - \delta u, \quad d = u - \alpha x.$$

Therefore,

$$\begin{array}{rcl} u & = & d + \alpha x, \\ z & = & c + \delta d + \delta\alpha x, \\ y & = & b + \gamma c + \gamma\delta d + \gamma\delta\alpha x; \\ x & = & a + \beta b + \beta\gamma c + \beta\gamma\delta d + \beta\gamma\delta\alpha x. \end{array}$$

The latter relation yields

$$(1 - \alpha\beta\gamma\delta)x = a + \beta b + \beta\gamma c + \beta\gamma\delta d.$$

Thus, if $1 - \alpha\beta\gamma\delta = 0$, then a relation of the form

$$a \pm b \pm c \pm d = 0$$

holds; it is also clear that the relation

$$a - b - c - d = 0$$

cannot be satisfied. Therefore, in our case $\alpha\beta\gamma\delta \neq 1$; hence, $\alpha\beta\gamma\delta = -1$. The numbers $\alpha, \beta, \gamma = \pm 1$ can be selected at random and the number δ is determined by these numbers.

There are altogether 8 distinct sets of numbers α, β, γ, δ and for each set there exists a unique solution x, y, z, u. Moreover, all the numbers x, y, z, u are nonzero and, therefore, all the 8 solutions are distinct.

b) **First solution.** Let us consider, for example, the case when

$$a + c = b + d, \text{ i.e., } a - b + c - d = 0.$$

In this case we have to set

$$\beta = -1, \ \beta\gamma = 1, \ \beta\gamma\delta = -1 \text{ and } \alpha\beta\gamma\delta = 1, \text{ i.e., } \alpha = \beta = \gamma = \delta = -1.$$

The system of equations for x, y, z, u considered in the solution of heading a) has infinitely many solutions:

$$u = d - x, \ z = c - d + x \text{ and } y = b - c + d - x = a - x,$$

where x is arbitrary.

Other cases are treated similarly: if

$$a + b = c + d,$$

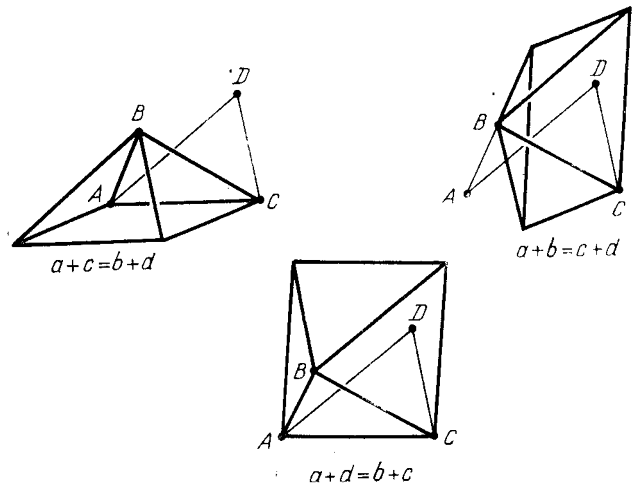

FIGURE 64 (SOL. 8.33)

then

$$\alpha = \gamma = -1 \text{ and } \beta = \delta = 1$$

and if

$$a + d = b + c,$$

then

$$\alpha = \gamma = 1 \text{ and } \beta = \delta = -1.$$

Second solution. In each of the three cases when the indicated relations hold we can construct a quadrilateral pyramid with vertex B whose lateral edges are equal and parallel to the sides of the given quadrilateral, the base is a parallelogram and the sum of the lengths of opposite edges are equal (see Fig. 64).

Therefore, there exists a ray with which the edges of the pyramid — hence, the sides of the quadrilateral – form equal angles (Problem 6.63). Let plane Π perpendicular to this ray intersect lines AB, BC, CD and DA at points P, Q, R and S, respectively, and the corresponding lateral edges of the pyramid at points P', Q', R' and S'. Since points P', Q', R' and S' lie on one circle and lines PQ and $P'Q'$, QR and $Q'R'$, etc., are parallel, it follows that

$$\angle(PQ, PS) = \angle(P'Q', P'S') = \angle(R'Q', R'S') = \angle(RQ, RS),$$

i.e., points P, Q, R and S lie on one circle (see \$); let O be the center of this circle. Since lines AP and AS form equal angles with plane Π, we deduce that $AP = AS$. It follows that the corresponding sides of triangles APO and ASO are equal and, therefore, the distances from point O to lines AB and AD are also equal.

We similarly prove that point O is equidistant from lines AB, BC, CD and DA, i.e., the sphere centered at O whose radius is equal to the distance from O to any of these lines is a desired one. By translating Π parallel to itself we get infinitely many such spheres.

REMARK. For every vertex of a spatial quadrilateral $ABCD$ we can consider two bisector planes that pass through the bisectors of its outer and inner angle perpendicularly to them. Clearly, O is the intersection point of bisector planes. The following quadruples of bisector planes intersect along one line:

all the 4 inner ones if $a + c = b + d$;

the inner ones at vertices A and C and outer ones at vertices B and D if $a+b = c + d$;

the inner ones at vertices B and D and outer ones at vertices A and C if $a+d = b + c$.

CHAPTER 9. REGULAR POLYHEDRONS

§1. Main properties of regular polyhedrons

A convex polyhedral angle is called a *regular* one if all its planar angles are equal and all the dihedral angles are also equal.

A convex polyhedron is called a *regular* one if all its faces and polyhedral angles are regular and, moreover, all the faces are equal and polyhedral angles are also equal. From the logic's point of view this definition is unsuccessful: it contains a lot of unnecessary. It would have been sufficient to require that the faces and the polyhedral angles were regular; this implies their equality. But such subtleties are not for the first acquaintance with regular polyhedrons. (We have devoted section 5 to the discussion of distinct equivalent definitions of regular polyhedrons.)

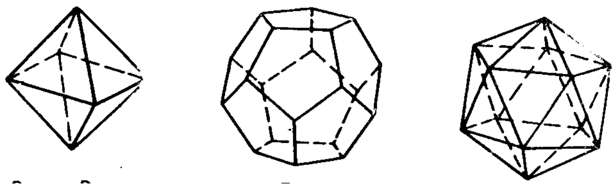

Figure 65 (§9)

There are only 5 distinct regular polyhedrons: *tetrahedron*, *cube*, *octahedron*, *dodecahedron* and *icosahedron*; the latter three polyhedrons are plotted on Fig. 65.

This picture does not, however, tell us much: it cannot replace neither the proof that there are no other regular polyhedrons nor even the proof of the fact that the regular polyhedrons plotted actually exist. (A picture can depict an optical illusion, cf. e.g., Escher's drawings.) All this is to be proved.

In one of the books that survived from antiquity to nowadays is written that octahedron and icosahedron were discovered by *Plato*'s student *Teatet* (410–368 B.C.) whereas cube, tetrahedron and dodecahedron were known to Pythagoreans long before him. Many of historians of mathematics doubted the truthfulness of these words; special incredulity were attributed to the fact that octahedron was discovered later than dodecahedron. Really, the Egyptian pyramids were constructed in ancient times and by joining mentally two pyramids we easily get an octahedron.

More accurate study, however, forces us to believe the words of the antient book. These words can hardly be interpreted otherwise as follows: Teatet distinguished a class of regular polyhedrons, i.e., with certain degree of rigor defined them, thus discovering their common property and proved that there are only 5 distinct types of regular polyhedrons.

Cube, tetrahedron and dodecahedron drew attention of geometers even before Teatet but only as simple and interesting geometric objects, not as regular polyhedrons. The ancient Greek terminology testifies the interest to cube, tetrahedron and dodecahedron: these polyhedrons had special names.

It is not wonder that cube and tetrahedron were always of interest to geometers; dodecahedron requires some elucidation. Crystals of pyrite encountered in nature have a shape close to that of dodecahedron. There survived also a dodecahedron manufactured for unknown purposes by Etruskians around 500 B.C.

The form of dodecahedron is incomparably more attractive and mysterious than the form of an octahedron. We think that dodecahedron should have intrigued Pythagoreans because a regular 5-angled star that one can naturally inscribe in every face of a dodecahedron was their symbol.

In the study of regular polyhedrons it is octahedron and icosahedron that cause the most serious troubles. By connecting three regular triangles, or three squares, or three regular pentagons and by continuing such construction we finally get a regular tetrahedron, cube or dodecahedron; at every stage we get a rigid construction.

For an octahedron or icosahedron we have to connect 4 or 5 triangles, respectively, i.e., the initial construction might collapse.

9.1. Prove that there can be no other regular polyhedrons except the above listed ones.

9.2. Prove that there exists a dodecahedron — a regular polyhedron with pentagonal faces and trihedral angles at vertices.

9.3. Prove that all the angles between nonparallel lines of a dodecahedron are equal.

9.4. Prove that there exists an icosahedron — a regular polyhedron with trihedral faces and 5-hedral angles at vertices.

9.5. Prove that for any regular polyhedron there exist:

a) a spere that passes through all its vertices (the *circumscribed sphere*);

b) a sphere tangent to all its faces (the *inscribed sphere*).

9.6. Prove that the center of the circumscribed sphere of a regular polyhedron is its center of mass (i.e., the center of mass of the system of points with unit masses at its vertices).

The center of the circumscribed sphere of a regular polyhedron that coincides with the center of the inscribed sphere and the center of mass, is called the *center* of the regular polyhedron.

§2. Relations between regular polyhedrons

9.7. a) Prove that it is possible to select 4 vertices of the cube so that they would be vertices of a regular tetrahedron. In how many ways can this be performed?

b) Prove that it is possible to select 4 planes of the faces of the octahedron so that they would be planes of faces of a regular tetrahedron. In how many ways can this be done?

9.8. Prove that on the edges of the cube one can select 6 points so that they will be vertices of an octahedron.

9.9. a) Prove that it is possible to select 8 vertices of the dodecahedron so that they will be vertices of a cube. In how many ways can this be done?

b) Prove that it is possible to select 4 vertices of a dodecahedron so that they will be vertices of a regular tetrahedron.

9.10. a) Prove that it is possible to select 8 planes of faces of an icosahedron so that they will be the planes of the faces of an octahedron. In how many ways can this be done?

b) Prove that it is possible to select 4 planes of the faces of an icosahedron so that they will be the planes of the faces of a regular tetrahedron.

* * *

9.11. Consider a convex polyhedron whose vertices are the centers of faces of the regular polyhedron. Prove that this polyhedron is also a regular one. (This polyhedron is called the polyhedron *dual* to the initial one).

9.12. a) Prove that the dual to the tetrahedron is a tetrahedron.

b) Prove that cube and octahedron are dual to each other.

c) Prove that dodecahedron and icosahedron are dual to each other.

9.13. Prove that if the radii of the inscribed spheres of two dual to each other regular polyhedrons are equal, then a) the radii of their circumscribed spheres are equal; b) the radii of circumscribed spheres of their faces are equal.

9.14. A face of a dodecahedron and a face of an icosahedron lie in one plane and, moreover, their opposite faces also lie in one plane. Prove that all the other vertices of the dodecahedron and icosahedron lie in two planes parallel to these faces.

§3. Projections and sections of regular polyhedrons

9.15. Prove that the projections of a dodecahedron and an icosahedron to planes parallel to their faces are regular polygons.

9.16. Prove that the projection of a dodecahedron to a plane perpendicular to the line that passes through its center and the midpoints of an edge is a hexagon (and not a decagon).

9.17. a) Prove that the projection of an icosahedron to the plane perpendicular to a line that passes through its center and a vertex is a regular decagon.

b) Prove that the projection of a dodecahedron to a plane perpendicular to a line that passes through its center and a vertex is an irregular dodecagon.

* * *

9.18. Is there a section of a cube which is a regular hexagon?

9.19. Is there a section of an octahedron which is a regular hexagon?

9.20. Is there a section of a dodecahedron which is a regular hexagon?

9.21. Faces ABC and ABD of an icosahedron have a common edge, AB. Through vertex D the plane is drawn parallel to plane ABC. Is it true that the section of the icosahedron with this plane is a regular hexagon?

§4. Self-superimpositions (symmetries) of regular polyhedrons

A motion that turns the polyhedron into itself (i.e., a symmetry) will be called a *self-superimposition.*

9.22. Which regular polyhedrons have a center of symmetry?

9.23. A convex polyhedron is symmetric relative a plane. Prove that either this plane passes through the midpoint of its edge or is the plane of symmetry of one of the polyhedral angles at its vertex.

9.24. a) Prove that for any regular polyhedron the planes passing through the midpoints of its edges perpendicularly to them are the planes of symmetry.

b) Which regular polyhedrons have in addition to the above other planes of symmetry?

9.25. Find the number of planes of symmetry of each of the regular polyhedrons.

9.26. Prove that any axis of rotation of a regular polyhedron passes through its center and either a vertex, or the center of an edge, or the center of a face.

9.27. a) How many axes of symmetry has each of the regular polyhedrons?

b) How many other axes of rotation has each of the regular polyhedrons?

9.28. How many self-superimpositions are there for each of the regular polyhedrons?

§5. Various definitions of regular polyhedrons

9.29. Prove that if all the faces of a convex polyhedron are equal regular polygons and all its dihedral angles are equal, then this polyhedron is a regular one.

9.30. Prove that if all the polyhedral angles of a convex polyhedron are regular ones and all its faces are regular polygons, then this polyhedron is a regular one.

9.31. Prove that if all the faces of a convex polyhedron are regular polygons and the endpoints of the edges that go out of every vertex form a regular polygon, then this polyhedron is a regular one.

* * *

9.32. Is it necessary that a convex polyhedron all faces of which and all the polyhedral angles of which are equal is a regular one?

9.33. Is it necessary that a convex polyhedron which has equal a) all the edges and all the dihedral angles; b) all the edges and all the polyhedral angles is a regular one?

Solutions

9.1. Consider an arbitrary regular polyhedron. Let all its faces be regular n-gons and all the polyhedral angles contain m faces each. Each edge connects two vertices and from every vertex m edges go out. Therefore, $2E = mV$. Similarly, every edge belongs to two faces and each face has n edges each. Therefore, $2E = nF$. Substituting these expressions into Euler's formula $V - E + F = 2$ (see Problem 8.14) we get $\frac{2}{m}E - E + \frac{2}{n}E = 2$, i.e.,

$$\frac{1}{n} + \frac{1}{m} = \frac{1}{2} + \frac{1}{E} > \frac{1}{2}.$$

Therefore, either $n < 4$ or $m < 4$. Thus, one of the numbers m and n is equal to 3; let the other number be equal to x. Now, we have to find all the integer solutions of the equation

$$\frac{1}{3} + \frac{1}{x} = \frac{1}{2} + \frac{1}{E}.$$

It is clear that $x = 6\frac{E}{E+6} < 6$, i.e., $x = 3, 4, 5$. Thus, there are only 5 distinct pairs of numbers (m, n):

1) (3, 3); the corresponding polyhedron is tetrahedron; it has 6 edges, 4 faces and 4 vertices;

2) (3, 4); the corresponding polyhedron is cube, it has 12 edges, 6 faces and 8 vertices;

3) (4, 3); the corresponding polyhedron is octahedron. It has 12 edges, 8 faces and 6 vertices;

4) (3, 5); the corresponding polyhedron is dodecahedron, it has 30 edges, 12 faces and 20 vertices;

5) (5, 3); the corresponding polyhedron is icosahedron. It has 30 edges, 20 faces and 12 vertices.

The number of edges, faces and vertices here were computed according to the formulas

$$\frac{1}{n}+\frac{1}{m}=\frac{1}{2}+\frac{1}{E},\ F=\frac{2}{n}E \text{ and } V=\tfrac{2}{m}E.$$

Remark. The polyhedrons of each of the above described type are determined uniquely up to similarity. Indeed, with the help of a similarity transformation we can identify a pair of faces of two polyhedrons of the same type so that the polyhedrons lie on one side of the plane of the identified faces. If the polyhedral angles are equal, then, as is easy to verify, the polyhedrons coincide.

The equality of the polyhedral angles is obvious for the trihedral angles, i.e., for tetrahedron, cube and dodecahedron. For the octahedron and icosahedron we can identify the polyhedrons dual to them; hence, the initial polyhedrons are also equal (cf. Problems 9.5, 9.11 and 9.12).

9.2. Proof is based on the properties of the figure that consists of three equal regular pentagons with a common vertex every two of which have a common edge.

In the solution of Problem 7.33 it was proved that the segments depicted on Fig. 53 by solid lines constitute a right trihedral angle, i.e., the considered figure can be applied to a cube so that these segments coincide with the cube's edges that go out of one vertex (Fig. 66). Let us prove that the obtained figure can be complemented to a dodecahedron with the help of symmetries through the planes parallel to the cube's faces and passing through its center.

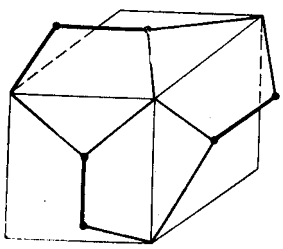

Figure 66 (Sol. 9.2)

The sides of a pentagon parallel to the edges of the cube are symmetric through the indicated planes. Besides, the distances from each of these sides to the face of the cube with which it is connected by three segments are equal (they are equal to $\sqrt{a^2-b^2}$, where a is the length of the segment that connects the vertex of the regular pentagon with the midpoint of the neighbouring side, b is a half length of the diagonal of the cube's face). Therefore, with the help of the indicated symmetries the considered figure can actually be complemented to a polyhedron. It remains to show that this polyhedron is a regular one, i.e., the dihedral angles at edges p_i that go out of the vertices of the cube are equal to the dihedral angles at edges q_j parallel to the faces of the cube.

To this end consider the symmetry through the plane that passes through the midpoint of edge p_i perpendicularly to it. This symmetry sends edge q_j that goes out of the second endpoint of edge p_i and is parallel to a face of the cube to edge p_k that goes out of a vertex of the cube.

9.3. For the neighbouring faces this statement is obvious. If F_1 and F_2 are non-neighbouring faces of the dodecahedron, then the face parallel to F_1 will be neighbouring to F_2.

9.4. Let us construct an icosahedron by arranging its vertices on the edges of an octahedron. Let us place arrows on the edges of the octahedron as shown on Fig. 67 a). Now, let us divide all the edges in the same ratio $\lambda : (1-\lambda)$ taking into account their orientation. The obtained points are vertices of a convex polyhedron with dihedral faces and 5-hedral angles at the vertices (Fig. 67 b)). Therefore, it suffices to select λ so that this polyhedron were a regular one.

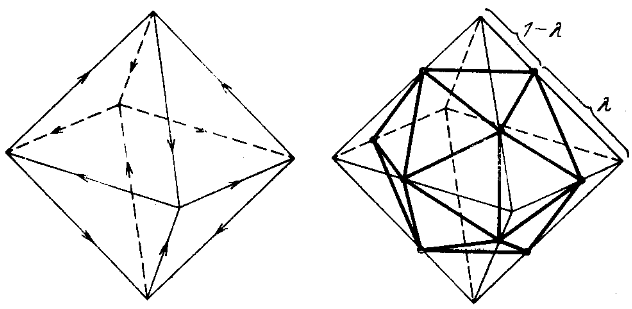

FIGURE 67 (SOL. 9.4)

It has two types of edges: those that belong to the faces of the octahedron and those that do not belong to them. The squared length of any edge that belongs to a face of the octahedron is equal to

$$\lambda^2 + (1-\lambda)^2 - 2\lambda(1-\lambda)\cos 60^\circ = 3\lambda^2 - 3\lambda + 1$$

and the squared length of any edge that does not belong to the face of the octahedron is equal to

$$2(1-\lambda)^2 = 2 - 4\lambda + 2\lambda^2.$$

(To prove the latter equality we have to take into account that the angle between non-neighbouring edges of the octahedron that exit one vertex is equal to 90°.)

Therefore, if $3\lambda^2 - 3\lambda + 1 = 2 - 4\lambda + 2\lambda^2$, i.e., $\lambda = \frac{\sqrt{5}-1}{2}$ (for obvious reasons we disregard the negative root), then all the faces of the obtained polyhedron are regular triangles. It remains to show that all the dihedral angles at its edges are equal. This easily follows from the fact that (for any λ) the vertices of the obtained polyhedron are equidistant from the center of the octahedron, i.e., belong to a sphere.

9.5. Let us draw perpendiculars to all the faces through their centers. It is easy to see that for two neighbouring faces such perpendiculars intersect at one point whose distance from each of the faces is equal to $a\cot\varphi$, where a is the distance from the center of the face to its sides and φ is a half of the dihedral angle between the faces of the polyhedron.

To this end we have to consider the section that passes through the centers of two neighbouring faces and the midpoint of their common edge (Fig. 68). Thus, on each of our perpendiculars we can mark a point and for neighbouring faces these points coincide. Therefore, all these perpendiculars have a common point O.

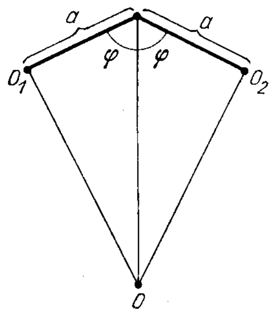

FIGURE 68 (SOL. 9.5)

It is clear that the distance from O to each vertex of the polyhedron is equal to $a/\cos\varphi$ and the distance to each face is equal to $-a\cot\varphi$, i.e., point O serves as the center of the circumscribed as well as the center of the inscribed sphere.

9.6. We have to show that the sum of vectors that connect the center of the circumscribed sphere of the regular polyhedron with its vertices is equal to zero. Denote this sum by $\mathbf{x}$. Any rotation that identifies the polyhedron with itself preserves the center of the inscribed sphere and, therefore, sends vector $\mathbf{x}$ into itself.

But a nonzero vector can only pass into itself under a rotation about an axis parallel to it. It remains to notice that any regular polyhedron has several axes the rotations about which turn it into itself.

9.7. a) If $ABCDA_1B_1C_1D_1$ is a cube, then AB_1CD_1 and A_1BC_1D are regular tetrahedrons.

b) It is easy to verify that the midpoints of the edges of a regular tetrahedron are vertices of an octahedron. This shows that we can select 4 faces of an octahedron so that they were planes of faces of a regular tetrahedron; one can do this in two ways.

9.8. Let the edge of cube $ABCDA_1B_1C_1D_1$ be of length $4a$. On the edges that exit vertex A, take points distant from it by $3a$. Similarly, take 3 points on the edges that exit vertex C_1. Making use of the identity

$$3^2 + 3^2 = 1 + 4^2 + 1$$

it is easy to verify that the lengths of all edges of the polyhedron with vertices in the selected points are equal to $3\sqrt{2}a$.

9.9. a) It is clear from the solution of Problem 9.2 that there exists a cube whose vertices are in the vertices of a dodecahedron. On each face of the dodecahedron there is a vertex of a cube. It is also clear that choosing for an edge of the cube any of the 5 diagonals of a face of the dodecahedron we uniquely fix the whole cube. Therefore, there are 5 distinct cubes with vertices in vertices of the dodecahedron.

b) Placing the cube so that its vertices are in vertices of the dodecahedron we can then place a regular tetrahedron so that its vertices are in vertices of this cube.

9.10. a) It is clear from the solution of Problem 9.4 that one can select 8 faces of an icosahedron so that they are faces of an octahedron. Then for every vertex of the icosahedron there exists exactly one edge (having that vertex as an endpoint) that does not lie in the plane of the face of the octahedron. It is also clear that the selection of any of the 5 edges that go out of the vertex of the icosahedron is the edge that does not belong to the plane of the octahedron's face uniquely determines the octahedron. Therefore, there are 5 distinct octahedrons the planes of whose faces pass through the faces of the icosahedron.

b) Selecting 8 planes of the icosahedron's faces so that they are also planes of an octahedron's faces we can select from them 4 planes so that they are planes of a regular tetrahedron's faces.

9.11. Consider the line that connects a vertex of the initial polyhedron with its center. The rotation about this line under which the polyhedron is sent into itself sends the centers of faces adjacent to the vertex mentioned above into themselves, i.e., these centers are vertices of a regular polyhedron.

Similarly, consider the line connecting the center of a face of the initial polyhedron with its center. A rotation about this line demonstrates that the polyhedral angles of the dual polyhedron are also regular ones. Since any two polyhedral angles of the initial polyhedron can be identified by a motion, all the faces of the dual polyhedron are equal. And since any two faces of the initial polyhedron can be identified, all the polyhedral angles of the dual polyhedron are equal.

9.12. To prove this statement, it suffices to notice that if the initial polyhedron has m-hedral angles at vertices and n-gonal faces, then the dual polyhedron has n-hedral angles at vertices and m-gonal faces.

Remark. The solutions of Problems 9.2 and 9.4 are, actually, two distinct solutions of the same problem. Indeed, if there exists a dodecahedron then there exists the polyhedron dual to it — an icosahedron; and the other way round.

9.13. a) Let O be the center of the initial polyhedron, A one of its vertices, B the center of one of the faces with vertex A. Consider the face of the dual polyhedron formed by the centers of the faces of the initial polyhedron adjacent to vertex A. Let C be the center of this face, i.e., the intersection point of this face with line OA.

Clearly, $AB \perp OB$ and $BC \perp OA$. Therefore, $OC : OB = OB : OA$, i.e., $r_2 : R_2 = r_1 : R_1$, where r_1 and R_1 (resp. r_2 and R_2) are the radii of the inscribed and circumscribed spheres of the initial polyhedron (resp. its dual).

b) If the distance from the plane to the center of the sphere of radius R is equal to r, then the plane cuts on the sphere a circle of radius $\sqrt{R^2 - r^2}$. Therefore, the radius of the circumscribed circles of the faces of the polyhedron inscribed into the sphere of radius R and circumscribed about the sphere of radius r is equal to $\sqrt{R^2 - r^2}$. In particular, if the values of R and r are equal for two polyhedrons, then the radii of the circumscribed circles of their faces are also equal.

9.14. If the dodecahedron and the icosahedron are inscribed in one sphere, then the radii of their inscribed spheres are equal (Problem 9.13 a), i.e., the distances between their opposite faces are equal. For a dodecahedron (or an icosahedron) we will call the intersection point of the circumscribed sphere with the line that passes through its center and the center of one of its faces the *center of a spherical face* of the dodecahedron (icosahedron).

Fix one of the centers of the spherical faces of the dodecahedron and consider the distance from it to the vertices; among these distances there are exactly four distinct ones. To solve the problem, it suffices to show that this set of four distinct distances coincides with a similar set for the icosahedron.

It is easy to verify that the centers of spherical faces of the dodecahedron are the vertices of an icosahedron and the centers of spherical faces of the obtained icosahedron are the vertices of the initial dodecahedron. Therefore, any distance between the center of a spherical face and a vertex of the dodecahedron is the distance between a vertex and the center of a spherical face of an icosahedron.

9.15. To prove the statement, it suffices to notice that these polyhedrons are sent into themselves under the rotation that identifies the projection of the upper face with the projection of the lower face. Thus, the projection of the dodecahedron is a decagon that is sent into itself under a rotation by 36° (Fig. 69 a)) and the projection of the icosahedron is a hexagon that is sent into itself under the rotation by 60° (Fig. 69 b)).

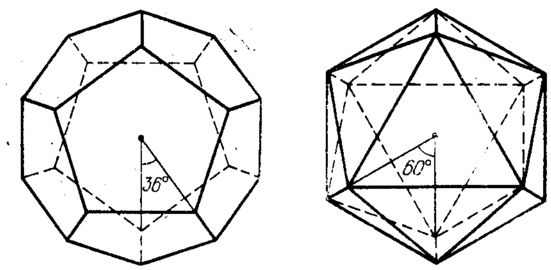

FIGURE 69 (SOL. 9.15)

9.16. Consider a cube whose vertices are in vertices of the dodecahedron (cf. Problem 9.2). In our problem we are talking about the projection to the plane parallel to a face of this cube. Now, it is easy to see that the projection of the dodecahedron is indeed a hexagon (Fig. 70).

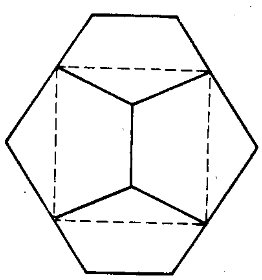

FIGURE 70 (SOL. 9.16)

9.17. a) The considered projection of icosahedron turns into itself under the rotation by 36° (this rotation sends the projections of the upper faces into the

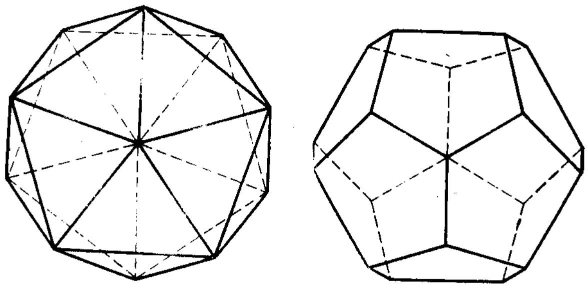

FIGURE 71 (SOL. 9.17)

projections of the lower faces). Therefore, this projection is a regular decagon (Fig. 71 a)).

b) The considered projection of the dodecahedron is a dodecagon that turns into itself under the rotation through an angle of 60° (Fig. 71 b)). A half of its sides are the projections of edges parallel to the plane of the projection and the other half of its sides are the projections of edges not parallel to the plane of the projection. Therefore, this dodecagon is an irregular one.

9.18. Yes, there is. The midpoints of the edges of the cube indicated by thick dots on Fig. 72 are the vertices of a regular hexagon. This follows from the fact that every side of this hexagon is parallel to a side of an equilateral triangle PQR and its length is equal to half the length of that triangle's side

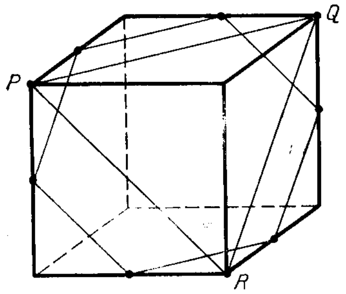

FIGURE 72 (SOL. 9.18)

9.19. There exists. Let us draw the plane parallel to two opposite faces of an octahedron and equidistant from them. It is easy to verify that the section with this plane is a regular hexagon (on Fig. 73 the projection onto this plane is depicted).

9.20. There exists. Take three pentagonal faces with common vertex A and consider the section with the plane that intersects these faces and is parallel to the plane in which three pairwise common vertices of the considered faces lie (Fig. 74). This section is a hexagon with pairwise parallel opposite sides.

After a rotation through an angle of 120° about the axis that passes through vertex A perpendicularly to the intersecting plane the dodecahedron and the intersecting plane turn into themselves.

Therefore, the section is a convex hexagon with angles 120° each the lengths of whose sides take two alternating values. In order for this hexagon to be regular

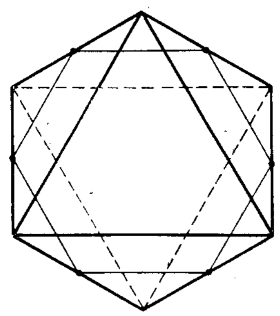

FIGURE 73 (SOL. 9.19)

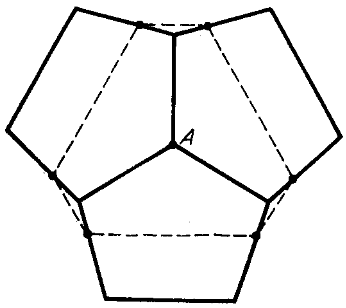

FIGURE 74 (SOL. 9.20)

it suffices for these two values to be equal. As the intersecting plane moves from one of its extreme positions to another one while moving away from vertex A, the first of these values grows from 0 to d while the second one diminishes from d to a, where a is the length of the dodecahedron's edge and d is the length of its face's diagonal ($d > a$). Therefore, at some moment these values become equal, i.e., the section is a regular hexagon.

9.21. No, this is false. Consider the projection of the icosahedron to plane ABC. It is a regular hexagon (cf. Problem 9.15 and Fig. 69). Therefore, the considered section is a regular hexagon only if all the 6 vertices connected by edges with points A, B and C (and distinct from A, B and C) lie in one plane. But it is easy to see that this is false (otherwise the vertices of the icosahedron would have lain on three parallel planes).

9.22. It is easy to verify that all the regular polyhedrons, except tetrahedron, have a center of symmetry.

9.23. A plane of symmetry divides a polyhedron into two parts and, therefore, it intersects at least one edge. Let us consider two cases.

1) The plane of symmetry passes through a vertex of the polyhedron. Then it is a plane of symmetry of the polyhedral angle at this vertex.

2) The plane of symmetry passes through an inner point of an edge. Then this edge turns into itself under the symmetry through this plane, i.e., the plane passes through the midpoint of the edge perpendicularly to it.

9.24. a) For the tetrahedron, cube and octahedron the statement of the problem is obvious. For the dodecahedron and icosahedron we have to make use of solutions of Problems 9.2 and 9.4, respectively. In doing so it is convenient to consider for the dodecahedron the plane that passes through the midpoint of an edge parallel to the cube's face and for the icosahedron a plane that passes through the midpoint of an edge that does not lie in the plane of the octahedron's face.

b) We have to find out for which polyhedral angles of regular polyhedrons there exist planes of symmetry that do not pass through the midpoints of edges. For a tetrahedron, dodecahedron and icosahedron,0 any plane of symmetry of a polyhedral angle does pass through the midpoints of its edges. For a cube and an octahedron there are planes of symmetry of polyhedral angles that do not pass through the midpoints of edges. These planes pass through the pairs of opposite edges.

9.25. First, let us consider the planes of symmetry that pass through the midpoints of edges perpendicularly to them. We have to find out through how many midpoints such a plane passes simultaneously.

It is easy to verify that for the tetrahedron each plane passes through the midpoint of one edge for the ocahedron, dodecahedron and icosahedron through the midpoints of two edges, and for the cube through the midpoints of 4 edges. Therefore, the number of such planes for the tetrahedron is equal to 4 for the cube it is equal to $\frac{12}{4} = 3$, for the octahedron to $\frac{12}{2} = 6$ and for the dodecahedron and icosahedron it is equal to $\frac{30}{2} = 15$.

The cube and the octahedron have another planes of symmetry as well; these planes pass through the pairs of opposite edges and for the cube such a plane passes through 2 edges, for the octahedron it passes through 4 edges. Therefore, the number of such planes for the cube is equal to $\frac{12}{2} = 6$ and for the octahedron it is equal to $\frac{12}{4} = 3$. Altogether the cube and the octahedron have 9 planes of symmetry each.

9.26. An axis of rotation intersects the surface of the polyhedron at two points. Let us consider one of these points. Three variants are possible:

1) The point is a vertex of the polyhedron.

2) The point belongs to an edge of the polyhedron but is not its vertex. Then this edge turns into itself under a rotation about it. Therefore, this point is the midpoint of the edge and the angle of the rotation is equal to 180°.

3) The point belongs to a face of the polyhedron but does not belong to an edge. Then this face turns into itself under a rotation and, therefore, this point is the center of the face.

9.27. a) For every regular polyhedron the lines that pass through the midpoints of opposite edges are the axes of symmetry. There are 3 such axes in a tetrahedron; 6 in a cube and an octahedron; 15 in a dodecahedron and icosahedron. Moreover, in the cube the lines that pass through the centers of faces and in the octahedron the lines that pass through vertices are axes of symmetry; there are 3 such axes for each of these polyhedrons.

b) A line will be called an *axis of rotation of order* n (for the given figure) if after the rotation through an angle of $\frac{2\pi}{n}$ the figure turns into itself. The lines that pass through vertices and the centers of faces of tetrahedron are axes of order 3; there are 4 such axes.

The lines that pass through the pairs of vertices of cube are axes of order 3; there are 4 such axes. The lines that pass through the pairs of centers of faces of the cube are axes of order 4; there are 3 such axes.

The lines that pass through the pairs of centers of faces of the octahedron are axes of order 3; there are 4 such axes. The lines that pass through the pairs of vertices of the octahedron are axes of order 4; there are 3 such axes.

The lines that pass through the pairs of vertices of the dodecahedron are axes of order 3; there are 10 such axes. The lines that pass through the pairs of centers of faces of the dodecahedron are axes of order 5; there are 6 such axes.

The lines that pass through the pairs of centers of faces of the icosahedron are axes of order 3; there are 10 such axes. The lines that pass through the pairs of vertices of the icosahedron are axes of order 5; there are 6 such axes.

9.28. Any face of a regular polyhedron can be transported by a motion into any other face. If the faces of a polyhedron are n-gonal ones, then there are exactly $2n$ motions that identifies the polyhedron with itself and preserves one of the faces: n rotations and n symmetries through planes. Therefore, the number of motions (the identical transformation included) is equal to $2nF$, where F is the number of faces.

Thus, the number of motions of the tetrahedron is equal to 24, that of the cube and octahedron is equal to 48, that of the dodecahedron and the icosahedron is equal to 120.

Remark. By similar arguments we can show that the number of motions of a regular polyhedron is equal to the doubled product of the number of its vertices by the number of faces of its polyhedral angles.

9.29. We have to prove that all the polyhedral angles of our polyhedron are equal. But its dihedral angles are equal by the hypothesis and planar angles are the angles of equal polygons.

9.30. We have to prove that all the faces are equal and the polyhedral angles are also equal. First, let us prove the equality of faces. Let us consider all the faces at a vertex. The polyhedral angle of this vertex is a regular one and, therefore, all its planar angles are equal, hence, all the angles of the considered regular polygons are also equal. Moreover, all the sides of the regular polygons with a common side are equal. Therefore, all the considered polygons are equal; hence, all the faces of the polyhedron are equal.

Now, let us prove that the polyhedron angles are equal. Let us consider all the polyhedral angles at vertices of one of the faces. One of the plane angles of each of them is the angle of this face and, therefore, all the plane angles of the considered polyhedral angles are equal. Moreover, the polyhedral angles with vertices are the endpoints of one edge have a common dihedral angle, hence, all their dihedral angles are equal. Therefore, all the considered polyhedral angles are equal; consequently, all the polyhedral angles of our polyhedron are equal.

9.31. We have to prove that all the polyhedral angles of our polyhedron are right ones. Let us consider the endpoints of all the edges that exit a vertex. As follows from the hypothesis of the problem, the polyhedron with vertices at these points and at point A is a pyramid whose ase is a regular polygon and all the edges of this pyramid are equal.

Therefore, point A belongs to the intersection of the planes that pass through the midpoints of the sides of the base perpendicular to them, i.e., it lies on the perpendicular to the base passing through the center of the base. Therefore, the pyramid is a regular one; it follows that the polyhedral angle at its vertex is a regular one.

9.32. No, not necessarily. Let us consider a (distinct from a cube) rectangular parallelepiped $ABCDA_1B_1C_1D_1$. In tetrahedron AB_1CD_1 all the faces and the trihedral angles are equal but it is not a regular one.

9.33. No, not necessarily. Let us consider the convex polyhedron whose vertices are the midpoints of cube's edges. It is easy to verify that all the edges, all the dihedral angles and all the polyhedral angles of this polyhedron are equal.

CHAPTER 10. GEOMETRIC INEQUALITIES

§1. Lengths, perimeters

10.1. Let a, b and c be the lengths of sides of a parallelepiped, d that of its of its diagonals. Prove that
$$a^2+b^2+c^2\geq \frac{d^2}{3}.$$

10.2. Given a cube with edge 1, prove that the sum of distances from an arbitrary point to all its vertices is no less than $4\sqrt{3}$.

10.3. In tetrahedron $ABCD$ the planar angles at vertex A are equal to 60°. Prove that
$$AB+AC+AD\leq BC+CD+DB.$$

10.4. From points A_1, A_2 and A_3 that lie on line a perpendiculars A_iB_i are dropped to line b. Prove that if point A_2 lies between A_1 and A_3 then the length of segment A_2B_2 is confined between the lengths of segments A_1B_1 and A_3B_3.

10.5. A segment lies inside a convex polyhedron. Prove that the segment is not longer than the longest segment with the endpoints at vertices of the polyhedron.

10.6. Let P be the projection of point M to the plane that contains points A, B and C. Prove that if one can construct a triangle from segments PA, PB and PC, then from segments MA, MB and MC one can also construct a triangle.

10.7. Points P and Q are taken inside a convex polyhedron. Prove that one of the vertices of the polyhedron is closer to Q than to P.

10.8. Point O lies inside tetrahedron $ABCD$. Prove that the sum of the lengths of segments OA, OB, OC and OD does not exceed the sum of the lengths of tetrahedron's edges.

10.9. Inside the cube with edge 1 several segments lie and any plane parallel to one of the cube's faces does not intersect more than one segment. Prove that the sum of the lengths of these segments does not exceed 3.

10.10. A closed broken line passes along the surface of a cube with edge 1 and has common points with all the cube's faces. Prove that its length is no less than $3\sqrt{2}$.

10.11. A tetrahedron inscribed in a sphere of radius R contains the center of the sphere. Prove that the sum of the lengths of the tetrahedron's edges is greater than $6R$.

10.12. The section of a regular tetrahedron is a quadrilateral. Prove that the perimeter of this quadrilateral is confined between $2a$ and $3a$, where a is the length of the tetrahedron's edge.

§2. Angles

10.13. Prove that the sum of the angles of a spatial quadrilateral does not exceed 360°.

10.14. Prove that not more than 1 vertex of a tetrahedron has a property that the sum of any two of plane angles at this vertex is greater than 180°.

10.15. Point O lies on the base of triangular pyramid $SABC$. Prove that the sum of the angles between ray SO and the lateral edges is smaller than the sum of the plane angles at vertex S while being greater than half this sum.

10.16. a) Prove that the sum of the angles between the edges of a trihedral angle and the planes of the faces opposite to them does not exceed the sum of its plane angles.

b) Prove that if dihedral angles of a trihedral angle are acute ones then the sum of the angles between its edges and planes of faces opposite to them is not less than a half sum of its plane angles.

10.17. The diagonal of a rectangular parallelepiped constitutes angles α, β and γ with its edges. Prove that $\alpha + \beta + \gamma < \pi$.

10.18. All the plane angles of a convex quadrangular angle are equal to 60°. Prove that the angles between its opposite edges cannot be neither simultaneously acute nor simultaneously obtuse.

10.19. Prove that the sum of all the angles that have a common vertex inside a tetrahedron and subtend the edges of that tetrahedron is greater than 3π.

10.20. a) Prove that the sum of dihedral angles at edges AB, BC, CD and DA of tetrahedron $ABCD$ is smaller than 2π.

b) Prove that the sum of dihedral angles of a tetrahedron is confined between 2π and 3π.

10.21. The space is completely covered by a finite set of (infinite one way) right circular coni with angles $\varphi_1, \dots , \varphi_n$. Prove that

$$\varphi_1^2 + \dots + \varphi_n^2 \geq 16.$$

§3. Areas

10.22. Prove that the area of any face of a tetrahedron is smaller(?) than the sum of the areas of its other three faces.

10.23. A convex polyhedron lies inside another polyhedron. Prove that the surface area of the outer polyhedron is greater than the surface area of the inner one.

10.24. Prove that for any tetrahedron there exist two planes such that the ratio of the areas of the tetrahedron's projections to them is not less than $\sqrt{2}$.

10.25. a) Prove that the area of any triangular section of a tetrahedron does not exceed the area of one of the tetrahedron's faces.

b) Prove that the area of any quadrangular section of a tetrahedron does not exceed the area of one of the tetrahedron's faces.

10.26. A plane tangent to the sphere inscribed in a cube cuts off it a triangular pyramid. Prove that the surface area of this pyramid does not exceed the area of the cube's face.

§4. Volumes

10.27. On each edge of a tetrahedron a point is fixed. Consider four tetrahedrons one of the vertices of each of which is a vertex of the initial tetrahedron and the remaining vertices are fixed points belonging to the edges that go out of this vertex. Prove that the volume of one of the tetrahedrons does not exceed $\frac{1}{8}$ of the initial tetrahedron's volume.

10.28. The lengths of each of the 5 edges of a tetrahedron do not exceed 1. Prove that its volume does not exceed $\frac{1}{8}$.

10.29. The volume of a convex polyhedron is equal to V and its surface area is equal to S.

a) Prove that if a sphere of radius r is placed inside the polyhedron, then $\frac{V}{S} \geq \frac{r}{3}$.

b) Prove that a sphere of radius $\frac{V}{S}$ can be placed inside the polyhedron.

c) A convex polyhedron is placed inside another one. Let V_1 and S_1 be the volume and the surface area of the outer polyhedron, V_2 and S_2 same of the outer one. Prove that

$$\frac{3V_1}{S_1} \geq \frac{V_2}{S_2}.$$

10.30. Inside a cube, a convex polyhedron is placed whose projection onto each face of the cube coincides with this face. Prove that the volume of the polyhedron is not less than $\frac{1}{3}$ the volume of the cube.

10.31. The areas of the projections of the body to coordinate axes are equal to S_1, S_2 and S_3. Prove that its volume does not exceed $\sqrt{S_1S_2S_3}$.

§5. Miscellaneous problems

10.32. Prove that the radius of the inscribed circle of any face of a tetrahedron is greater than the radius of the sphere inscribed in the tetrahedron.

10.33. On the base of a triangular pyramid $OABC$ with vertex O point M is taken. Prove that

$$OM \cdot S_{ABC} \leq OA \cdot S_{MBC} + OB \cdot S_{MAC} + OC \cdot S_{MAB}.$$

10.34. Let r and R be the radii of the inscribed and circumscribed spheres of a regular quadrangular pyramid. Prove that

$$\frac{R}{r} \geq 1 + \sqrt{2}.$$

10.35. Is it possible to cut a hole in a cube through which another cube of the same size can be pulled?

10.36. Sections M_1 and M_2 of a convex centrally symmetric polyhedron are parallel and M_1 passes through the center of symmetry.

a) Is it true that the area of M_1 is not less than the area of M_2?

b) Is it true that the radius of the minimal circle that contains M_1 is not less than the radius of the minimal circle that contains M_2?

10.37. A convex polyhedron sits inside a sphere of radius R. The length of its i-th edge is equal to l_i and the dihedral angle at this edge is equal to φ_i. Prove that

$$\sum l_i(\pi - \varphi_i) \leq 8\pi R.$$

Problems for independent study

10.38. Triangle $A'B'C'$ is a projection of triangle ABC. Prove that the hights of triangle $A'B'C'$ are no longer than the corresponding hights of triangle ABC.

10.39. A sphere is inscribed into a truncated cone. Prove that the surface area of the ball is smaller than the area of the lateral surface of the cone.

10.40. The largest of the perimeters of tetrahedron's faces is equal to d and the sum of the lengths of its edges is equal to D. Prove that

$$3d < 2D \leq 4d.$$

10.41. Inside tetrahedron $ABCD$ a point E is fixed. Prove that at least one of segments AE, BE and CE is shorter than the corresponding segment AD, BD and CD.

10.42. Is it possible to place 5 points inside a regular tetrahedron with edge 1 so that the pairwise distances between these points would be not less than 1?

10.43. The plane angles of a trihedral angle are α, β and γ. Prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma \leq 1 + 2\cos\alpha\cos\beta\cos\gamma.$$

10.44. The base of pyramid $ABCDE$ is a parallelogram $ABCD$. None of the lateral faces is an acute triangle. On edge DC, there is a point M such that line EM is perpendicular to BC. Moreover, diagonal AC of the base and lateral edges ED and EB are connected by relations $AC \geq \frac{5}{4}EB \geq \frac{5}{3}ED$. Through vertex B and the midpoint of one of lateral edges a section is drawn; the section is an isosceles trapezoid. Find the ratio of the area of the section to the area of the pyramid's base.

Solutions

10.1. Since $d \leq a + b + c$, it follows that

$$d^2 \leq a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \leq 3(a^2 + b^2 + c^2).$$

10.2. If PQ is the diagonal of cube with edge 1 and X is an arbitrary point, then $PX + QX \geq PQ = \sqrt{2}$. Since cube has 4 diagonals, the sum of the distances from X to all the vertices of the cube is not less than $4\sqrt{3}$.

10.3. First, let us prove that if $\angle BAC = 60°$, then $AB + AC \leq 2BC$. To this end let us consider points B' and C' symmetric to points B and C through the bisector of angle A. Since in any convex quadrilateral the sum of the lengths of diagonals is greater than the sum of the lengths of a pair of opposite sides,

$$BC + B'C' \geq CC' + BB'$$

(the equality is attained if $AB = AC$). It remains to notice that $B'C' = BC$, $CC' = AC$ and $BB' = AB$.

We similarly prove inequalities $AC + AD \leq 2CD$ and $AD + AB \leq 2DB$. By adding up these inequalities we get the desired statement.

10.4. Let us draw through line b a plane Π parallel to a. Let C_i be the projection of point A_i to plane Π. By the theorem on three perpendiculars, $C_iB_i \perp b$; therefore, the length of segment B_2C_2 is confined between the length of B_1C_1 and that of B_3C_3; the lengths of all three segments A_iC_i are equal.

10.5. In the proof we will several times make use of the following planimetric statement:

If point X lies on side BC of triangle ABC, then either $AB \geq AX$ or $AC \geq AX$.

(Indeed, one of the angles BXA or CXA is not less than $90°$; if $\angle BXA \geq 90°$, then $AB \geq AX$ and if $\angle CAX \geq 90°$, then $AC \geq AX$.)

Let us extend the given segment to its intersection with the polyhedron's faces at certain points P and Q; this might only increase the length of the segment. Let MN be an arbitrary segment with the endpoints on the edges of the polyhedron; let P belong to MN. Then either $MQ \geq PQ$ or $NQ \geq PQ$.

Let, for definiteness, $MQ \geq PQ$. Point M lies on an edge AB and either $AQ \geq MQ$ or $BQ \geq MQ$. We have replaced segment PQ by a longer segment one of whose endpoints lies in a vertex of the polyhedron. Now, performe similar argument for the endpoint Q of the obtained segment. We can replace PQ by a longer segment with the endpoints in vertices of the polyhedron.

10.6. Let $a = PA$, $b = PB$ and $c = PC$. We can assume that $a \leq b \leq c$. Then by the hypothesis $c < a + b$. Further, let $h = PM$. We have to prove that

$$\sqrt{c^2 + h^2} < \sqrt{a^2 + h^2} + \sqrt{b^2 + h^2},$$

i.e.,

$$c\sqrt{1 + \left(\frac{h}{c}\right)^2} < a\sqrt{1 + \left(\frac{h}{a}\right)^2} + b\sqrt{1 + \left(\frac{h}{b}\right)^2}.$$

It remains to notice that

$$c\sqrt{1 + \left(\frac{h}{c}\right)^2} < (a+b)\sqrt{1 + \left(\frac{h}{c}\right)^2} \leq a\sqrt{1 + \left(\frac{h}{a}\right)^2} + b\sqrt{1 + \left(\frac{h}{b}\right)^2}.$$

10.7. Let us consider plane Π that passes through the midpoint of segment PQ perpendicularly to it. Suppose that all the vertices of the polyhedron are not closer to point Q than to point P. Then all the vertices of the polyhedron lie on the same side of plane Π as point P does. Therefore, point Q lies outside the polyhedron which contradicts the hypothesis.

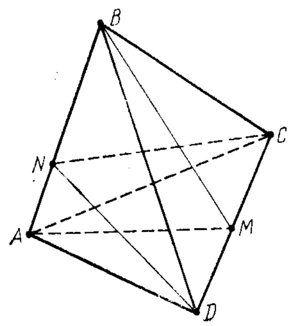

FIGURE 75 (SOL. 10.8)

10.8. Let M and N be the intersection points of planes AOB and COD with edges CD and AB, respectively (Fig. 75). Since triangle AOB lies inside triangle AMB, it follows that

$$AO + BO \leq AM + BM.$$

Similarly,

$$CO + DO \leq CN + DN.$$

Therefore, it suffices to prove that the sum of the lengths of segments AM, BM, CN and DN does not exceed the sum of the lengths of the edges of tetrahedron $ABCD$.

First, let us prove that if X is a point on side $A'B'$ of triangle $A'B'C'$, then the length of segment $C'X$ does not exceed a semi-perimeter of triangle $A'B'C'$. Indeed,

$$C'X \leq C'B' + B'X \text{ and } C'X \leq C'A + A'X.$$

Therefore,

$$2C'X \leq A'B' + B'C' + C'A'.$$

Thus,

$$\begin{aligned} 2AM &\leq AC + CD + DA, 2BM \leq BC + CD + DB, \\ 2CN &\leq BA + AC + CB,\ 2DN \leq BA + AD + DB. \end{aligned}$$

By adding up all these inequalities we get the desired statement.

10.9. Let us enumerate the segments and consider the i-th segment. Let l_i be its length, x_i, y_i, z_i the lengths of projections on the cube's edges. It is easy to verify that $l_i \leq x_i + y_i + z_i$.

On the other hand, if any plane parallel to the cube's face intersects not more than 1 segment, then the projections of these segments to each edge of the cube do not have common points. Therefore, $\sum x_i \leq 1, \sum y_i \leq 1, \sum z_i \leq 1$ and, finally, $\sum l_i \leq 3$.

10.10. Consider the projections on 3 nonparallel edges of the cube. The projection of the given broken line on any edge contains both endpoints of the edge and, therefore, it coincides with the whole edge. Hence, the sum of the lengths of the projections of the broken line's links on any edge is no less than 2 and the sum of the lengths of projections on all the three edges is not less than 6.

One of the three lengths of projections of any broken line's link on the cube's edges is zero; let two other lengths of projections be equal to a and b. Since $(a + b)^2 \leq 2(a^2 + b^2)$, it follows that the sum of the lengths of the links of the broken line is no less than the sum of the lengths of these projections on the three edges of the cube divided by $\sqrt{2}$; hence, it is no less than $\frac{6}{\sqrt{2}} = 3\sqrt{2}$.

10.11. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ be vectors that go from the center of the sphere to the vertices of the tetrahedron. Since the center of the sphere lies inside the tetrahedron, there exist positive numbers $\lambda_1, \dots, \lambda_4$ such that

$$\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 + \lambda_4\mathbf{v}_4 = 0$$

(see Problem 7.16). We may assume that $\lambda_1 + \dots + \lambda_4 = 1$. Let us prove that then $\lambda_i \leq \frac{1}{2}$. Let, for example, $\lambda_1 > \frac{1}{2}$. Then

$$\frac{R}{2} < |\lambda_1\mathbf{v}_1| = |\lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 + \lambda_4\mathbf{v}_4| \leq (\lambda_2 + \lambda_3 + \lambda_4)R = (1 - \lambda_1)R < \frac{R}{2}.$$

We have got a contradiction because $\lambda_i \leq \frac{1}{2}$. Therefore,

$$\begin{aligned} |\mathbf{v}_1 + \dots + \mathbf{v}_4| &= |(1 - 2\lambda_1)\mathbf{v}_1 + \dots + (1 - 2\lambda_4)\mathbf{v}_4| \\ &\leq ((1 - 2\lambda_1) + \dots + (1 - 2\lambda_4))R = 2R. \end{aligned}$$

Since

$$\sum |\mathbf{v}_i - \mathbf{v}_j|^2 = (4R)^2 - |\sum \mathbf{v}_i|^2$$

(see the solution of Problem 14.15) and $|\sum \mathbf{v}_i|^2 \leq 2R$, it follows that

$$\sum |\mathbf{v}_i - \mathbf{v}_j|^2 \geq (16-4)R^2 = 12R^2.$$

And since $2R > |\mathbf{v}_i - \mathbf{v}_j|$, it follows that

$$2R \sum |\mathbf{v}_i - \mathbf{v}_j| > \sum |\mathbf{v}_i - \mathbf{v}_j|^2 \geq 12R^2.$$

10.12. Let us consider all the sections of the tetrahedron by the planes parallel to the given sections. Those of them that are quadrilaterals turn under the projection on the line perpendicular to the planes of the sections into the inner points of segment PQ, where points P and Q correspond to sections with planes passing through the vertices of the tetrahedron (Fig. 76 a)).

The length of the side of the section that belongs to a fixed face of the tetrahedron is a linear function on segment PQ. Therefore, the perimeter of the section being the sum of linear functions is a linear function on segment PQ. The value of a linear function at an arbitrary point of PQ is confined between its values at points P and Q.

Therefore, it suffices to verify that the perimeter of the section of a regular tetrahedron by a plane that passes through a vertex of the tetrahedron is confined between $2a$ and $3a$ (except for the cases when the section consists of one point; but such a section cannot correspond to neither P nor Q). If the section is an edge of the tetrahedron then the value of the considered linear function is equal to $2a$ for it.

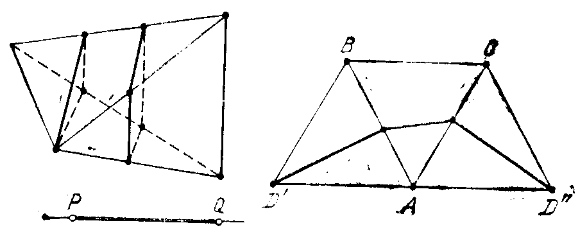

FIGURE 76 (SOL. 10.12)

Since the length of any segment with the endpoints on sides of an equilateral triangle does not exceed the length of this triangle's side, the perimeter of a triangular section of the tetrahedron does not exceed $3a$.

If the plane of the section passes through vertex D of tetrahedron $ABCD$ and intersects edges AB and AC, then we will unfold faces ABD and ACD to plane ABC (Fig. 76 b)). The sides of the section connect points D' and D'' and, therefore, the sum of their lengths is no less than $D'D'' = 2a$.

10.13. If the vertices of a spatial quadrilateral $ABCD$ are not in one plane, then

$$\angle ABC < \angle ABD + \angle DBC \text{ and } \angle ADC < \angle ADB + \angle BDC$$

(cf. Problem 5.4). Adding up these inequalities and adding further to both sides angles $\angle BAD$ and $\angle BCD$ we get the desired statement, because the sums of the angles of triangles ABD and DBC are equal to $180°$.

10.14. Suppose that vertices A and B of tetrahedron $ABCD$ have the indicated property. Then

$$\angle CAB + \angle DAB > 180^\circ \text{ and } \angle CBA + \angle DBA > 180^\circ.$$

On the other hand,

$$\angle CAB + \angle CBA = 180^\circ - \angle ACB < 180^\circ \text{ and } \angle DBA + \angle DAB < 180^\circ.$$

Contradiction.

10.15. By Problem 5.4 $\angle ASB < \angle ASO + \angle BSO$. Since ray SO lies inside the trihedral angle $SABC$, it follows that

$$\angle ASO + \angle BSO < \angle ASC + \angle BSC$$

(cf. Problem 5.6). By writing down two more pairs of such inequalities and taking their sum we get the desired statement.

10.16. a) Let α, β and γ be the angles between edges SA, SB and SC and the planes of the faces opposite to them, respectively. Since the angle between line l and plane Π does not exceed the angle between line l and any line in plane Π, it follows that

$$\alpha \leq \angle ASB, \beta \leq \angle BSC \text{ and } \gamma \leq \angle CSA.$$

b) The dihedral angles of the trihedral angle $SABC$ are all acute and, therefore, the projection SA_1 of ray SA to plane SBC lies inside angle BSC. Therefore, the inequalities

$$\angle ASB \leq \angle BSA_1 + \angle ASA_1 \text{ and } \angle ASC \leq \angle ASA_1 + \angle CSA_1$$

yield

$$\angle ASB + \angle ASC - \angle BSC \leq 2\angle ASA_1.$$

Write similar inequalities for edges SB and SC and take their sum. We get the desired statement.

10.17. Let O be the center of the rectangular parallelepiped $ABCDA_1B_1C_1D_1$. Height OH of an isosceles triangle AOC is parallel to edge AA_1 and, therefore, $\angle AOC = 2\alpha$, where α is the angle between edge AA_1 and diagonal AC_1. Similar arguments show that the plane angles of the trihedral angle $OACD_1$ are equal to 2α, 2β and 2γ. Therefore, $2\alpha + 2\beta + 2\gamma < 2\pi$.

10.18. Let S be the vertex of the given angle. From solutions of Problem 5.16 b) it follows that it is possible to intersect this angle with a plane so that in the section we get rhombus $ABCD$, where $SA = SC$ and $SB = SD$, and the projection of vertex S to the plane of the section coincides with the intersection point of the diagonals of the rhombus, O. Angle ASC is acute if $AO < SO$ and obtuse if $AO > SO$. Since $\angle ASB = 60^\circ$, it follows that

$$AB^2 = AS^2 + BS^2 - AS \cdot BS.$$

Expressing, thanks to Pythagoras theorem, AB, AS and BS via AO, BO and SO we get after simplification and squaring

$$(1 + a^2)(1 + b^2) = 4, \text{ where } a = \frac{AO}{SO} \text{ and } b = \frac{BO}{SO}.$$

Therefore, the inequalities $a > 1$ and $b > 1$, as well as inequalities $a < 1$ and $b < 1$, cannot hold simultaneously.

10.19. Let O be a point inside tetrahedron $ABCD$; let α, β and γ be angles with vertex O that subtend the edges AD, BD and CD; let a, b and c be angles with vertex O that subtend the edges BC, CA and AB; P the intersection point of line DO with face ABC. Since ray OP lies inside the trihedral angle $OABC$, it follows that

$$\angle AOP + \angle BOP < \angle AOC + \angle BOC$$

(cf. Problem 5.6), i.e., $\pi - \alpha + \pi - \beta < b + a$ and, therefore,

$$\alpha + \beta + a + b > 2\pi.$$

Similarly,

$$\beta + \gamma + b + c > 2\pi \text{ and } \alpha + \gamma + a + c > 2\pi.$$

Adding up these inequalities we get the desired statement.

10.20. a) Let us apply the statement of Problem 7.19 to tetrahedron $ABCD$. Let **a**, **b**, **c** and **d** be normal vectors to faces BCD, ACD, ABD and ABC, respectively. The sum of these vectors is equal to 0 and, therefore, there exists a spatial quadrilateral the vectors of whose consecutive sides are **a**, **b**, **c** and **d**.

The angle between sides **a** and **b** of this quadrilateral is equal to the dihedral angle at edge CD (cf. Fig. 77). Similar arguments show that the considered sum of the dihedral angles is equal to the sum of plane angles of the obtained quadrilateral which is smaller than 2π (Problem 10.13).

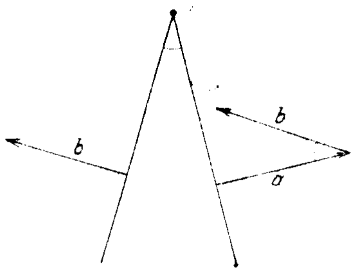

FIGURE 77 (SOL. 10.20)

b) Let us express the inequality obtained in heading a) for each pair of the opposite edges of the tetrahedron and add up these three inequalities. Each dihedral angle of the tetrahedron enters two such inequalities and, therefore, the doubled sum of the dihedral angles of the tetrahedron is smaller than 6π.

The sum of the dihedral angles of any trihedral angle is greater than π (Problem 5.5). Let us write such an inequality for each of the four vertices of the tetrahedron and add up these inequalities. Each dihedral angle of the tetrahedron enters two such inequalities (corresponding to the endpoints of an edge) and, therefore, the doubled sum of the dihedral angles of the tetrahedron is greater than 4π.

10.21. The vertices of all the coni can be confined in a ball of radius r. Consider a sphere of radius R with the same center O. As $\frac{R}{r}$ tends to infinity, the share of the surface of this sphere confined inside the given coni tends to the share of its surface confined inside the coni with the same angles, vertices at point O, and the axes parallel to the axes of the given coni.

Since the solid angle of the cone with angle φ is equal to $4\pi\sin^2\left(\frac{\varphi}{4}\right)$ (Problem 4.50), it follows that

$$4\pi\left(\sin^2\left(\frac{\varphi_1}{4}\right)+\dots+\sin^2\left(\frac{\varphi_n}{4}\right)\right)\geq 4\pi.$$

It remains to observe that $x\geq\sin x$.

10.22. For any tetrahedron the projections of its three faces on the plane of the remaining face completely cover that face. It is also clear that the area of the projection of a triangle on a plane not parallel to it is smaller than the area of the triangle itself (see Problem 2.13).

10.23. On faces of the inner polyhedron construct outwards, as on bases, rectangular prisms whose edges are sufficiently long: all of them should intersect the surface of the outer polyhedron. These prisms cut on the surface of the outer polyhedron pairwise nonintersecting figures, the area of each one of these being no less than that of the base of the prism, i.e. the area of a face of the inner polyhedron.

Indeed, the projection of each such figure on the plane of the base of the prism coincides with the base itself and the projection can only diminish the area of a figure.

10.24. Let plane Π be parallel to two skew edges of the tetrahedron. Let us prove that the desired two planes can be found even among the planes perpendicular to Π.

The projection of the tetrahedron on any such plane is a trapezoid (or a triangle) whose heights are equal to the distance between the chosen skew edges of the tetrahedron. The midline of this trapezoid (triangle) is the projection of a parallelogram with vertices at the midpoints of the four edges of the tetrahedron.

Therefore, it remains to verify that for any parallelogram there exist two lines (in the same plane) such that the ratio of the lengths of the projections of the parallelogram to them is not less than $\sqrt{2}$. Let a and b be the sides of parallelogram's sides ($a\leq b$) and d the length of its greatest diagonal. The length of the projection of the parallelogram to the line perpendicular to side b does not exceed a; the length of the projection to a line parallel to the diagonal d is equal to d. It is also clear that $d^2\geq a^2+b^2\geq 2a^2$.

10.25. a) If the triangular section does not pass through a vertex of the tetrahedron, then there exists a parallel to it triangular section that does pass through a vertex; the area of the latter section is greater.

Therefore, it suffices to consider cases when the section passes through a vertex or an edge of the tetrahedron.

Let point M lie on edge CD of tetrahedron $ABCD$. The length of the height dropped from point M to line AB is confined between the lengths of heights dropped to this line from points C and D (Problem 10.4). Therefore, either $S_{ABM}\leq S_{ABC}$ or $S_{ABM}\leq S_{ABD}$.

Let points M and N lie on edges CD and CB respectively of tetrahedron $ABCD$. To section AMN of tetrahedron $AMBC$ we can apply the statement just proved. Therefore, either $S_{AMN}\leq S_{ACM}\leq S_{ACD}$ or $S_{AMN}\leq S_{ABM}$.

b) Let the plane intersect edges AB, CD, BD and AC of tetrahedron $ABCD$ at points K, L, M and N, respectively. Let us consider the projection to the plane perpendicular to line MN (Fig. 78 a)). Since $K'L'=KL\sin\varphi$, where φ is the angle

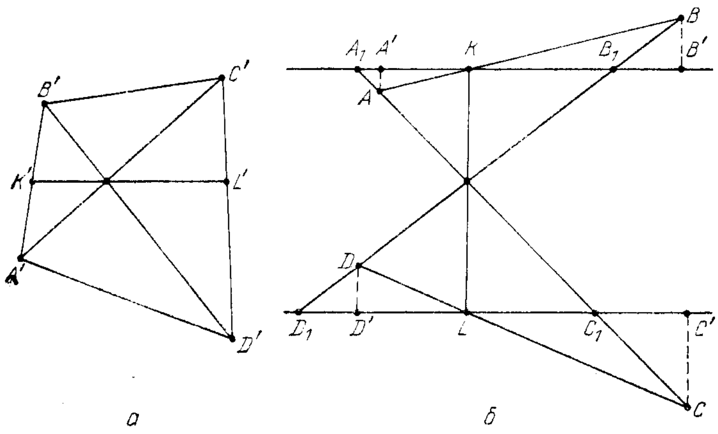

FIGURE 78 (SOL. 10.25)

between lines KL and MN, we see that the area of the section of the tetrahedron is equal to $K'L' \cdot \frac{MN}{2}$. Therefore, it suffices to prove that either $K'L' \leq A'C'$ or $K'L' \leq B'D'$.

It remains to prove the following planimetric statement:

The length of segment KL that passes through the intersection point of diagonals of convex quadrilateral $ABCD$ does not exceed the length of one of its diagonals (the endpoints of the segment lie on sides of the quadrilateral).

Let us draw lines through the endpoints of segment KL perpendicular to it and consider the projections on KL of vertices of the quadrilateral and the intersection points of lines AC and BD with the perpendiculars to KL we erected (Fig. 78 b)).

Let, for definiteness, point A lie inside the strip given by these lines and point B be outside it. Then we may assume that D lies inside the strip because otherwise $BD > KL$ and the proof is completed. Since

$$\frac{AA'}{BB'} \leq \frac{AK}{BK} = \frac{C_1L}{D_1L} \leq \frac{CC'}{DD'},$$

it follows that either $AA' \leq CC'$ (and, therefore, $AC > KL$) or $BB' \geq DD'$ (and, therefore, $BD > KL$).

10.26. Let the given plane intersect edges AB, AD and AA' at points K, L and M, respectively; let P, Q and R be the centers of faces $ABB'A'$, $ABCD$ and $ADD'A'$, respectively; let O be the tangent point of the plane with the sphere.

Planes KOM and KPM are tangent to the sphere at points O and P and, therefore, $\angle KOM = \angle KPM$. Hence, $\angle KOM = \angle KPM$. Similar arguments show that

$$\angle KPM + \angle MRL + \angle LQK = \angle KOM + \angle MOL + \angle LOK = 360^\circ.$$

It is also clear that $KP = KQ$, $LQ = LR$ and $MR = MP$; hence, quadrilaterals $AKPM$, $AMRL$ and $ALQK$ can be added as indicated on Fig. 79.

In hexagon ALA_1MA_2K the angles at vertices A, A_1 and A_2 are right ones and, therefore,

$$\angle K + \angle L + \angle M = 4\pi = 1.5\pi = 2.5\pi$$

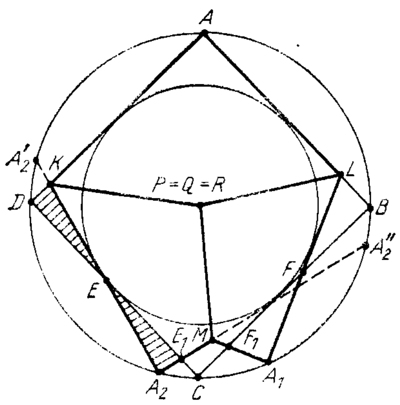

FIGURE 79 (SOL. 10.26)

and since angles K, L and M are greater than $\frac{\pi}{2}$, it follows that two of them, say, K and L, are smaller than π. These argument show that point A_2 lies on arc $\smile DC$, A_1 on arc $\smile CB$ and, therefore, point M lies inside square $ABCD$.

The symmetry through the midperpendicular to segment DA_2 sends both circles into themselves and, therefore, the tangent lines DA and DC turn into A_2A_2'' and A_2A_2'. Hence, $\triangle DKE = \triangle A_2E_1E$. Similarly, $\triangle BLF = \triangle A_1F_1F$. Therefore, the area of hexagon ALA_1MA_2K, being equal to the surface area of the given pyramid, is smaller than the area of square $ABCD$.

10.27. If two tetrahedrons have a common trihedral angle, then the ratio of their volumes is equal to the product of the ratios of the lengths of edges that lie on the edges of this trihedral angle (cf. Problem 3.1).

Therefore, the product of the ratios of volumes of the considered four tetrahedrons to the volume of the initial one is equal to the product of numbers of the form $A_iB_{ij} : A_iA_j$, where A_i and A_j are vertices of the tetrahedron, B_{ij} is a point fixed on edge A_iA_j. To every edge A_iA_j there corresponds a pair of such numbers, $A_iB_{ij} : A_iA_j$ and $A_iB_{ij} : A_iA_j$. If $A_iA_j = a$ and $A_iB_{ij} = x$, then $A_jB_{ij} = a - x$. Therefore, the product of the pair of numbers corresponding to edge A_iA_j is equal to $\frac{x(a-x)}{a^2} \leq \frac{1}{4}$.

Since a tetrahedron has 6 edges, the considered product of the four ratios of volumes of tetrahedrons does not exceed $\frac{1}{4^6} = \frac{1}{8^4}$. Therefore, one of the ratios of volumes does not exceed $\frac{1}{8}$.

10.28. Let the lengths of all edges of tetrahedron $ABCD$, except for edge CD, do not exceed 1. If h_1 and h_2 are heights dropped from vertices C and D to line AB and $a = AB$, then the volume V of tetrahedron $ABCD$ is equal to $ah_1h_2 \sin \frac{1}{6}\varphi$, where φ is the dihedral angle at edge AB. In triangle with sides a, b and c, the squared length of the height dropped to a is equal to

$$\frac{b^2 - x^2 + c^2 - (a-x)^2}{2} \leq \frac{b^2 + c^2 - \frac{1}{2}a^2}{2}.$$

In our case $h^2 \leq 1 - \frac{a^2}{4}$, hence, $V \leq \frac{a(1-a^2/4)}{6}$, where $0 < a \leq 1$. By calculating the derivative of the function $a(1 - \frac{a^2}{4})$ we see that it grows monotonously from

0 to $\sqrt{\frac{4}{3}}$ and, therefore, so it does on the segment $[0, 1]$. At $a = 1$ the value of $\frac{1}{6}a(1 - a^2/4)$ is equal to $\frac{1}{8}$.

10.29. a) Let O be the center of the given sphere. Let us divide the given polyhedron into pyramids with vertex O whose bases are the faces of the polyhedron. The heights of these pyramids are no less than r and, therefore, (1) the sum of their volumes is not less than $\frac{Sr}{3}$, (2) $V \geq \frac{Sr}{3}$.

b) On the faces of the given polyhedron as on bases, construct inward rectangular prisms of height $h = \frac{V}{S}$. These prisms can intersect and go out of the polyhedron and the sum of their volumes is equal to $hS = V$; therefore, there remains a point of the polyhedron not covered by them. The sphere of radius $\frac{V}{S}$ centered at this point does not intersect the faces of the given polyhedron.

c) According to heading b) in an inner point of the polyhedron one can place a sphere of radius $r = \frac{V_2}{S_2}$ that does not intersect the faces of the given polyhedron. Since this sphere lies inside the outer polyhedron, then by heading a)

$$\frac{V_1}{S_1} \geq \frac{r}{3}.$$

10.30. On each edge of the cube there is a point of the polyhedron because otherwise its projection along this edge would not have coincided with the face. On each edge of the cube take a point of the polyhedron and consider the new convex polyhedron with vertices at these points. Since the new polyhedron is contained in the initial polyhedron, it suffices to prove that its volume is not less than $\frac{1}{3}$ of the volume of the cube.

We may assume that the length of the cube's edge is equal to 1. The considered polyhedron is obtained by cutting off tetrahedrons from the trihedral angles at the vertices of the cube. Let us prove that the sum of volumes of two tetrahedrons for vertices that belong to the same edge of the cube does not exceed $\frac{1}{6}$. This sum is equal to $\frac{1}{3}S_1h_1 + \frac{1}{3}S_2h_2$, where h_1 and h_2 are the heights dropped to the opposite faces of the cube from a vertex of the polyhedron that lies on the given edge of the cube and S_1 and S_2 are the areas of the corresponding faces of the tetrahedrons. It remains to observe that

$$S_1 \leq \frac{1}{2},\ S_2 \leq \frac{1}{2} \text{ and } h_1 + h_2 = 1.$$

Four parallel edges of the cube determine a partition of its vertices into 4 pairs. Therefore, the volume of all the cut off tetrahedrons does not exceed $\frac{4}{6} = \frac{2}{3}$, i.e., the volume of the remaining part is not less than $\frac{1}{3}$.

If $ABCDA_1B_1C_1D_1$ is the given cube, then the polyhedrons for which the equality is attained are tetrahedrons AB_1CD_1 and A_1BC_1D.

10.31. Let us draw planes parallel to coordinate planes and distant from them by $n\varepsilon$, where n runs over integers and ε is a fixed number. These planes divide the space into cubes with edge ε.

It suffices to carry out the proof for the bodies that consist of these cubes. Indeed, if we tend ε to zero then the volume and the areas of the projections of the body that consists of the cubes lying inside the initial body will tend to the volume and the area of the projections of the initial body.

First, let us prove that if the body is cut in two by a plane parallel to the coordinate plane and for both parts the indicated inequality holds, then it holds for the whole body. Let V be the volume of the whole body, S_1, S_2 and S_3 the areas of its projections on coordinate planes; the volume and the area of its first and second parts will be denoted by the same letters with one and two primes respectively.

We have to prove that the inequalities $V' \leq \sqrt{S_1'S_2'S_3'}$ and $V'' \leq \sqrt{S_1''S_2''S_3''}$ imply $V = V' + V'' \leq \sqrt{S_1S_2S_3}$. Since $S_3' \leq S_3$ and $S_3'' \leq S_3$, it suffices to verify that

$$\sqrt{S_1'S_2'} + \sqrt{S_1''S_2''} \leq \sqrt{S_1S_2}.$$

We may assume that S_3 is the area of the projection to the plane that cuts the body. Then $S_1 = S_1' + S_1''$ and $S_2 = S_2' + S_2''$. It remains to verify that

$$\sqrt{ab} + \sqrt{cd} \leq \sqrt{a + c)(b + d)}.$$

To prove this we have to square both parts and make use of the inequality

$$\sqrt{(ad)(bc)} \leq \frac{1}{2}(ad + bc).$$

The proof of the required inequality will be carried out by induction on the height of the body, i.e., on the number of layers of the cubes from which the body is composed. By the previous argument we have actually proved the inductive step. The base of induction, however, is not yet proved, i.e., we have not considered the case of the body that consists of one layer of cubes.

In this case we will carry out the proof again by induction with the help of the above proved statement: let us cut the body into rectangular parallelepipeds of size $\varepsilon \times \varepsilon \times n\varepsilon$.

The validity of the required inequality for one such parallelepiped, i.e., the base of induction, is easy to verify.

10.32. Let us consider the section of tetrahedron by the plane parallel to face ABC and passing through the center of its inscribed sphere. This section is triangle $A_1B_1C_1$ similar to triangle ABC and the similarity coefficient is smaller than 1. Triangle $A_1B_1C_1$ contains a circle of radius r, where r is the radius of the inscribed sphere of tetrahedron. Draw tangents parallel to sides of triangle $A_1B_1C_1$ to this circle; we get a still smaller triangle circumscribed about the circle of radius r.

10.33. Let $p = S_{MBC} : S_{ABC}$, $q = S_{MAC} : S_{ABC}$ and $r = S_{MAB} : S_{ABC}$. By Problem 7.12

$$\{OM\} = p\{OA\} + q\{OB\} + r\{OC\}$$

It remains to notice that

$$OM \leq pOA + qOB + rOC.$$

10.34. Let $2a$ be the side of the base of the pyramid, h its height. Then r is the radius of the circle inscribed in an isosceles triangle with height h and base $2a$; let R be the radius of the circumscribed circle of an isosceles triangle with height h and base $2\sqrt{2}a$. Therefore, $r(a + \sqrt{a^2 + h^2}) = ah$, i.e., $rh = a(\sqrt{a^2 + h^2} - a)$.

If b is a lateral side of an isosceles triangle, then $2R : b = b : h$, i.e., $2Rh = b^2 = 2a^2 + h^2$. Therefore,

$$k = \frac{R}{r} = \frac{2a^2 + h^2}{2a(\sqrt{a^2 + h^2} - a)},$$

i.e.,

$$(2a^2k + 2a^2 + h^2)^2 = 4a^2k^2(a^2 + h^2).$$

Let $x = \frac{h^2}{a^2}$, then

$$x^2 + 4x(1 + k - k^2) + 4 + 8k = 0.$$

The discriminant of this quadratic equation in x is equal to $16k^2(k^2 - 2k - 1)$. Since $k > 0$ and, therefore, this quadratic has real roots, it follows that $k \geq 1 + \sqrt{2}$.

10.35. This is possible. The projection of the cube with edge a to the plane perpendicular to the diagonal is a regular hexagon with side $b = \frac{a\sqrt{2}}{\sqrt{3}}$.

Let us inscribe in the obtained hexagon a square as plotted on Fig. 80. It is easy to verify that the side of this square is equal to $\frac{2\sqrt{3}b}{1+\sqrt{3}} = \frac{2\sqrt{2}a}{1+\sqrt{3}} > a$ and, therefore, it can contain inside itself a square K with side a. Cuttting a part of the cube whose projection is K we get the desired hole.

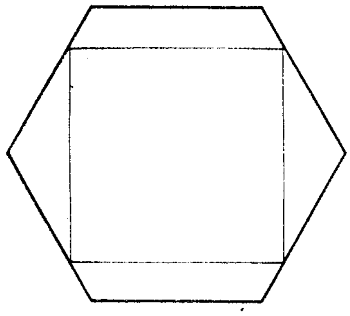

FIGURE 80 (SOL. 10.35)

10.36. a) Yes, this is true. Let O be the center of symmetry of the given polyhedron; M_2' the polygon symmetric to M_2 through point O. Let us consider the smallest (in area) convex polyhedron P that contains both M_2 and M_2'. Let us prove that the part of the area of section M_1 that lies inside P is not less than the area of M_2.

Let A be an inner point of a face N of polyhedron P distinct from M_2 and M_2' and let B be a point symmetric to A through O. A plane parallel to N intersects faces M_2 and M_2' only if it intersects segment AB; then it intersects M_1 as well.

Let the plane that passes through a point of segment AB parallel to face N intersect faces M_2 and M_2' along segments of length l and l', respectively; let it intersect the part of face M_1 that lies inside P along a segment of length m. Then $m \geq \frac{l}{2+l'}$ because polyhedron P is a convex one. Therefore, the area of M_1 is smaller than a half sum of the areas of M_2 and M_2', i.e., the area of M_2.

b) No, this is false. Let us consider a regular octahedron with edge a. The radius of the circumscribed circle of a face is equal to $\frac{a}{\sqrt{3}}$. A section parallel to a face and passing through the center of the octahedron is a regular hexagon with side $\frac{a}{2}$; the radius of its circumscribed circle is equal to $\frac{a}{2}$. Clearly, $\frac{a}{\sqrt{3}} > \frac{a}{2}$.

10.37. Let us consider the body that consists of points whose distance from the given polyhedron is $\leq d$. The surface area of this body is equal to

$$S + d\sum l_i(\pi - \varphi_i) + 4\pi d^2,$$

where S is the surface area of the polyhedron (Problem 3.13). Since this body is confined inside a sphere of radius $d + R$, the surface area of the body does not exceed $4\pi(d + R)^2$ (this statement is obtained by passage to the limit from the statement of Problem 10.23). Therefore,

$$S + d\sum l_i(\pi - \varphi_i) \leq 8\pi dR + 4\pi R^2.$$

By tending d to infinity we get the desired statement.

CHAPTER 11. PROBLEMS ON MAXIMUM AND MINIMUM

§1. A segment with the endpoints on skew lines

11.1. The endpoints of segment AB move along given lines a and b. Prove that the length of AB is the smallest possible when AB is perpendicular to both lines.

11.2. Find the least area of the section of a cube with edge a by a plane that passes through its diagonal.

11.3. All the edges of a regular triangular prism $ABCA_1B_1C_1$ are of length a. Points M and N lie on lines BC_1 and CA_1, so that line MN is parallel to plane AA_1B. When such a segment MN is the shortest?

11.4. Given cube $ABCDA_1B_1C_1D_1$ with edge a. The endpoints of a segment that intersects edge C_1D_1 lie on lines AA_1 and BC. What is the least length that this segment can have?

11.5. Given cube $ABCDA_1B_1C_1D_1$ with edge a. The endpoints of a segment that constitutes a 60° angle with the plane of face $ABCD$ lie on lines AB_1 and BC_1. What is the least length such a segment can have?

§2. Area and volume

11.6. What is the least value of the ratio of volumes of a cone and cylinder circumscribed about the same sphere?

11.7. The surface area of a spherical segment is equal to S (we have in mind only the spherical part of the surface). What is the largest possible volume of such a segment?

11.8. Prove that among all the regular n-gonal pyramids with fixed total area the pyramid whose dihedral angle at an edge of the base is equal to the dihedral angle at an edge of a regular tetrahedron has the largest volume.

11.9. Through point M inside a given trihedral angle with right planar angles all possible planes are drawn. Prove that the volume of a tetrahedron cut off such a plane from the trihedral angle is the least one when M is the intersection point of the medians of the triangle obtained in the section of the trihedral angle with this plane.

* * *

11.10. What is the greatest area of the projection of a regular tetrahedron with edge a to a plane?

11.11. What is the greatest area of the projection of a rectangular parallelepiped with edges a, b and c to a plane?

11.12. A cube with edge a lies on a plane. A source of light is situated at distance b from the plane, and $b > a$. Find the least value of the area of the shade the cube casts on the plane.

§3. Distances

11.13. a) For every inner point of a regular tetrahedron consider the sum of distances from the point to the vertices. Prove that the sum takes the least value for the center of the tetrahedron.

b) The lengths of two opposite edges of tetrahedron are equal to b and c that of the other edges are equal to a. What is the least value of the sum of distances from an arbitrary point in space to the vertices of this tetrahedron?

11.14. Given cube $ABCDA_1B_1C_1D_1$ with edge a. On rays A_1A, A_1B_1 and A_1D_1, points E, F and G, respectively, are taken such that $A_1E = A_1F = A_1G = b$. Let M be a point on circle S_1 inscribed in square $ABCD$ and N be a point on circle S_2 that passes through E, F and G. What is the least value of the length of segment MN?

11.15. In a truncated cone the angle between the axis and the generator is equal to 30°. Prove that the shortest way along the surface of the cone that connects a point on the boundary of one of the bases with the diametrically opposite point on the boundary of the other base is of length $2R$, where R is the radius of the greater base.

11.16. The lengths of three pairwise perpendicular segments OA, OB and OC are equal to a, b and c, respectively, where $a \leq b \leq c$. What is the least and greatest values that the sum of distances from points A, B and C to a line l that passes through O can take?

§4. Miscellaneous problems

11.17. Line l lies in the plane of one face of a given dihedral angle. Prove that the angle between l and the plane of the other face is the greatest when l is perpendicular to the edge of the given dihedral angle.

11.18. The height of a regular quadrangular prism $ABCDA_1B_1C_1D_1$ is two times shorter than the side of the base. Find the greatest value of angle A_1MC_1, where M is a point on edge AB.

11.19. Three identical cylindrical surfaces of radius R with mutually perpendicular axes are pairwise tangent to each other.

a) What is the radius of the smallest ball tangent to all these cylinders?

b) What is the radius of the largest cylinder tangent to the three given ones and whose axis passes inside the triangle with vertices at the tangent points of the given cylinders?

11.20. Can a regular tetrahedron with edge 1 fall through a circular hole of radius: a) 0.45; b) 0.44? (We ignore the thickness of the plane that hosts the hole).

Problems for independent study

11.21. What greatest volume can a quadrangular pyramid have if its base is a rectangular one side of which is equal to a and the lateral edges of the pyramid are equal to b?

11.22. What is the largest volume of tetrahedron $ABCD$ all vertices of which lie on a sphere of radius 1 and the center of the sphere is the vertex of angles of 60° that subtend edges AB, BC, CD and DA?

11.23. Two cones have a common base and are situated on different sides of it. The radius of the base is equal to r, the height of one of the cones is equal to h, that of another one is H $(h \le H)$. Find the greatest distance between two generators of these cones.

11.24. Point N lies on a diagonal of a lateral face of a cube with edge a, point M lies on the circle situated in the plane of the lower face of the cube and with the center at the center of this face. Find the least value of the length of segment MN.

11.25. Given a regular tetrahedron with edge a, find the radius of the ball centered in the center of the tetrahedron, for which the sum of the volumes of the part of the tetrahedron situated outside the ball and the part of the ball situated outside the tetrahedron takes the least value.

11.26. The diagonal of a unit cube lies on the edge of a dihedral angle of value α $(\alpha < 180^\circ)$. In what limits can the volume of the part of the cube confined inside the angle vary?

11.27. Two vertices of a tetrahedron lie on the surface of a sphere of radius $\sqrt{10}$ and two other vertices on the surface of the sphere of radius 2 concentric with the first one. What greatest volume can such a tetrahedron have?

11.28. The plane angles of one trihedral angle are equal to 60°, those of another one are equal to 90° and the distance between their vertices is equal to a; the vertex of each of them is equidistant from the faces of another one. Find the least value of their common part — the 6-hedron.

Solutions

11.1. Let us draw through line b a plane Π parallel to a. Let A' be the projection of point A to plane Π. Then

$$AB^2 = A'B^2 + A'A^2 = A'B^2 + h^2,$$

where h is the distance between line a and plane Π. Point A' coincides with B if $AB \perp \Pi$.

11.2. Let the plane pass through diagonal AC_1 of cube $ABCDA_1B_1C_1D_1$ and intersect its edges BB_1 and DD_1 at points P and Q, respectively. The area of the parallelogram APC_1Q is equal to the product of the length of segment AC_1 by the distance from point P to line AC_1. The distance from point P to line AC_1 is minimal when P lies on the common perpendicular to lines AC_1 and BB_1; the line that passes through the midpoints of edges BB_1 and DD_1 is this common perpendicular. Thus, the area of the section is the least one when P and Q are the midpoints of edges BB_1 and DD_1. This section is a rhombus with diagonals $AC_1 = a\sqrt{3}$ and $PQ = a\sqrt{2}$ and its area is equal to $\frac{a^2\sqrt{6}}{2}$.

11.3. If M' and N' are the projections of points M and N to plane ABC, then $M'N' \parallel AB$. Let $CM' = x$. Therefore, $M'N' = x$ and the length of the projection of segment MN to line CC_1 is equal to $|a - 2x|$. Hence,

$$MN^2 = x^2 + (a - 2x)^2 = 5x^2 - 4ax + a^2.$$

The least value of the length of segment MN is equal to $\frac{a}{\sqrt{5}}$.

11.4. Let points M and N lie on lines AA_1 and BC, respectively, and segment MN intersect edge C_1D_1 at point L. Then points M and N lie on rays AA_1 and BC so that $x = AM > a$ and $y = BN > a$. By considering the projections on planes AA_1B and ABC we get

$$C_1L : LD_1 = a : (x - a) \text{ and } C_1L : LD_1 = (y - a) : a.$$

respectively. Therefore, $(x-a)(y-a)=a^2$, i.e., $xy=(x+y)a$; hence, $(xy)^2=(x+y)^2a^2\geq 4xya^2$, i.e., $xy\geq 4a^2$. Therefore,

$$MN^2=x^2+y^2+a^2=(x+y)^2-2xy+a^2=\frac{(xy)^2}{a^2}-2xy+a^2=\frac{(xy-a^2)^2}{a^2}\geq 9a^2.$$

The least value of the length of segment MN is equal to $3a$; it is attained when $AM=BN=2a$.

11.5. Let us introduce a coordinate system directing axes Ox, Oy and Oz along rays BC, BA and BB_1, respectively. Let the coordinates of point M from line BC_1 be $(x,0,x)$ and those of point N from line B_1A be $(0,y,a-y)$. Then the squared length of segment MN is equal to $x^2+y^2(a-x-y)^2$ and the squared length of its projection M_1N_1 to plane of face $ABCD$ is equal to x^2+y^2. Since the angle between lines MN and M_1N_1 is equal to 60°, it follows that $MN=2M_1N_1$, i.e., $(a-x-y)^2=3(x^2+y^2)$.

Let $u^2=x^2+y^2$ and $v=x+y$. Then $MN=2M_1N_1=2u$. Moreover, $(a-v)^2=3u^2$ by the hypothesis and $2u^2\geq v^2$. Therefore, $(a-v)^2\geq\frac{3v^2}{2}$; hence, $v\leq a(\sqrt{6}-2)$. Therefore,

$$u^2=\frac{(a-v)^2}{3}\geq\frac{a^2(3-\sqrt{6})^2}{3}=a^2(\sqrt{3}-\sqrt{2})^2,$$

i.e., $MN\geq 2a(\sqrt{3}-\sqrt{2})$. The equality is attained when $x=y=\frac{a(\sqrt{6}-2)}{2}$.

11.6. Let r be the radius of the given sphere. If the axial section of the cone is an isosceles triangle with height h and base $2a$, then $ah=S=r(a+\sqrt{h^2+a^2})$. Therefore,

$$a^2(h-r)^2=r^2(h^2+a^2),\text{ i.e., } a^2=\tfrac{r^2h^2}{h-2r}.$$

Hence, the volume of the cone is equal to $\frac{\pi r^2h^2}{3(h-2r)}$. Since

$$\frac{d}{dh}\left(\frac{h^2}{h-2r}\right)=-\frac{4rh-h^2}{(h-2r)^2},$$

it follows that the volume of the cone is minimal at $h=4r$. In this case the ratio of volumes of the cone to the cylinder is equal to $\frac{4}{3}$.

11.7. Let V be the volume of the spherical segment, R the radius of the sphere. Since $S=2\pi Rh$ (by Problem 4.24) and $V=\frac{\pi h^2(3R-h)}{3}$ (by Problem 4.27), it follows that

$$V=\frac{Sh}{2}-\frac{\pi h^3}{3}.$$

Therefore, the derivative of V with respect to h is equal to $\frac{S}{2}-\pi h^2$. The greatest volume is attained at $h=\sqrt{\frac{S}{2\pi}}$; it is equal to $S\sqrt{\frac{S}{18\pi}}$.

11.8. Let h be the height of a regular pyramid, r the radius of the inscribed circle of its base. Then the volume and the total area of the pyramid's surface are equal to

$$\frac{n}{3}\tan\frac{\pi}{n}(r^2h)\text{ and } n\tan\tfrac{\pi}{n}(r^2+r\sqrt{h^2+r^2}),$$

respectively. Thus, the quantity

$$r^2 + r\sqrt{h^2 + r^2} = a$$

is fixed and we have to find out when the quantity r^2h attains the maximal value (it is already clear that the answer does not depend on n).

Since

$$h^2 + r^2 = (\frac{a}{r} - r)^2 = (\frac{a}{r})^2 - 2a + r^2,$$

it follows that

$$(r^2h)^2 = a^2r^2 - 2ar^4.$$

The derivative of this function with respect to r is equal to $2a^2r - 8ar^3$. Therefore, the volume of the pyramid is maximal if $r^2 = \frac{a}{4}$, i.e., $h^2 = 2a$. Therefore, if φ is the dihedral angle at an edge of the base of this pyramid, then $\tan^2\varphi = 8$, i.e., $\cos\varphi = \frac{1}{3}$.

11.9. Let us introduce a coordinate system directing its axes along the edges of the given trihedral angle. Let the coordinates of point M be (α, β, γ). Let the plane intersect the edges of the trihedral angle at points distant from its vertex by a, b and c. Then the equation of this plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Since the plane passes through point M, we have

$$\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 1.$$

The volume of the cutoff tetrahedron is equal to $\frac{abc}{6}$. The product abc takes the least value when the value of $\frac{\alpha\beta\gamma}{abc}$ is the greatest, i.e., when $\frac{\alpha}{a} = \frac{\beta}{b} = \frac{\gamma}{c} = \frac{1}{3}$.

11.10. The projection of a tetrahedron can be a triangle or a quadrilateral. In the first case it is the projection on one of the faces and, therefore, its area does not exceed $\frac{\sqrt{3}a^2}{4}$.

In the second case the diagonals of the quadrilateral are projections of the tetrahedron's edges and, therefore, the area of the shade, being equal to one half the product of the diagonal's lengths by the sine of the angle between them, does not exceed $\frac{a^2}{2}$.

The equality is attained when the pair of opposite edges of the terahedron is parallel to the given plane. It remains to notice that $\frac{\sqrt{3}a^2}{4} < \frac{a^2}{2}$.

11.11. The area of the projection of the parallelepiped is twice the area of the projection of one of the triangles with vertices at the endpoints of the three edges of the parallelepiped that exit one point; for example, if the projection of the parallelepiped is a hexagon then for such a vertex we should take a vertex whose projection lies inside the hexagon.

For a rectangular parallelepiped all such triangles are equal. Therefore, the area of the projection of the parallelepiped is the greatest when one of these triangles is parallel to the plane of the projection. The greatest value is equal to $\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$ (cf. Problem 1.22).

11.12. Let $ABCD$ be a square with side a; let the distance from point X to line AB be equal to b, where $b > a$; let C' and D' be intersection points of the extensions of segments XC and XD beyond points C and D respectively with line AB. Since $\triangle C'D'X \sim \triangle CDX$, it follows that $x : b = a : (b - a)$, where $x = C'D'$. Therefore, $x = \frac{ab}{b-a}$. These arguments show that the area casted by the upper face of the cube is always a square of side $\frac{ab}{b-a}$.

Therefore, the area of the shade casted by the cube is the least when this shade coincides with the shade casted by the upper face only, i.e., when the source of light is placed above the upper face. But then the area of the shade is equal to $\left(\frac{ab}{a-b}\right)^2$ and the lower face of the cube is considered to be in the shade.

11.13. a) Through vertices of regular tetrahedron $ABCD$ let us draw planes parallel to its opposite faces. These planes also form a regular tetrahedron. Therefore, the sum of distances from those planes to an inner point X of tetrahedron $ABCD$ is constant (Problem 8.1 a)). The distance from point X to such a plane does not exceed the distance from point X to the corresponding vertex of the tetrahedron and the sum of distances from point X to the vertices of the tetrahedron is equal to the sum of distances from point X to these planes only if X is the center of tetrahedron.

b) In tetrahedron $ABCD$, let the lengths of edges AB and CD be equal to b and c respectively and the length of the other edges be equal to a. If M and N be the midpoints of edges AB and CD respectively, then line MN is an axis of symmetry for tetrahedron $ABCD$. Let X be an arbitrary point in space; point Y be symmetric to it through line MN; let K the midpoint of segment XY (it lies on line MN). Then

$$XA + XB = XA + YA \geq 2KA = KA + KB.$$

Similarly,

$$XC + XD \geq KC + KD.$$

Therefore, it suffices to find out what is the least value of the sum of distances from the vertices of the tetrahedron to a point on line MN.

For the points of this line the sum of distances to the vertices of the tetrahedron $ABCD$ does not vary if we rotate segment AB about this line so that it becomes parallel to CD. We then get an isosceles trapezoid $ABCD$ with bases b and c and height $MN = \sqrt{\frac{a^2-(b^2+c^2)}{4}}$.

For any convex quadrilateral the sum of distances from the vertices takes the least value at the intersection point of the diagonals; then it is equal to the sum of the diagonal's lengths. It is easy to verify that the sum of the diagonal's lengths of the obtained trapezoid $ABCD$ is equal to $\sqrt{4a^2 + 2bc}$.

11.14. Let O be the center of the cube. Consider two spheres with center O that contain circles S_1 and S_2, respectively. Let R_1 and R_2 be radii of these spheres.

The distance between points of circles S_1 and S_2 cannot be less than $|R_1 - R_2|$. If two cones with a common vertex O passing through S_1 and S_2, respectively, intersect (i.e., have a common generator), then the distance between S_1 and S_2 is equal to $|R_1 - R_2|$. If these cones do not intersect, then the distance between S_1 and S_2 is equal to the least of the distances between their points that lie in the plane that passes through point O and the centers of the circles, i.e., in plane AA_1CC_1. Let KL be the diameter of circle S_1 that lies in this plane; P the intersection point of lines OK and AA_1 (Fig. 81).

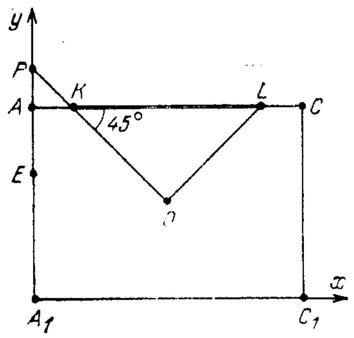

FIGURE 81 (SOL. 11.14)

Let us introduce a coordinate system directing axes Ox and Oy along rays A_1C_1 and A_1A. Points E, O and K have coordinates $(0, b)$, $(\frac{a}{\sqrt{2}}, \frac{a}{2})$ and $(\frac{a(\sqrt{2}-1)}{2}, a)$, respectively; therefore,

$$R_2 = OE = \sqrt{b^2 - ab + \frac{3a^2}{4}}; \quad EK = \sqrt{4b^2 - 8ab + \frac{(7-2\sqrt{2})a^2}{2}}.$$

It is also clear that $R_1 = \frac{a}{\sqrt{2}}$.

The cones intersect if $b = A_1E \geq A_1P = \frac{a(\sqrt{2}+1)}{2}$. In this case the least value of the length of MN is equal to $R_2 - R_1$. If $b < \frac{a(\sqrt{2}+1)}{2}$, then the cones do not intersect and the least value of the length of MN is equal to the length of EK.

11.15. Let us prove that the shortest way from point A on the boundry of the greatest base to the diametrically opposite point C of the other base is the union of the generator AB and diameter BC; the length of this pass is equal to $2R$. Let r be the radius of the smaller base, O its center. Let us consider a pass from point A to a point M of the smaller base.

Since the unfolding of the lateral surface of the cone with angle α between the axis and a generator is a sector of a circle of radius R with the length of the arc $2\pi R \sin\alpha$ then the unfolding of the lateral surface of this truncated cone with angle $\alpha = 30°$ is a half ring (annulus) with the outer radius $2R$ and the inner radius $2r$.

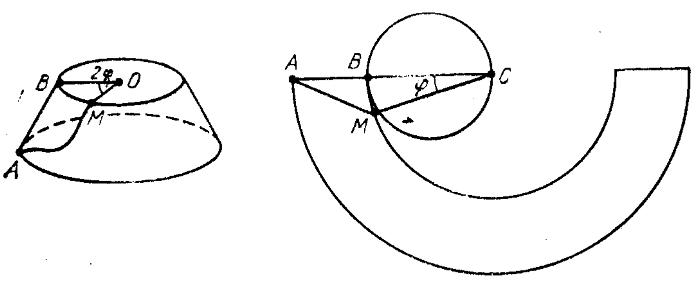

FIGURE 82 (SOL. 11.15)

Moreover, if $\angle BOM = 2\varphi$, then, on the unfolding, $\angle BCM = \varphi$ (cf. Fig. 82). The length of any pass from A to M is not shorter than the length of segment AM on the unfolding of the cone. Therefore, the length of a pass from A to C is not shorter than $AM + CM$, where

$$AM^2 = AC^2 + CM^2 - 2AM \cdot CM \cos ACM = 4R^2 + 4r^2 - 8Rr \cos\varphi$$

(on the unfolding) and

$$CM = 2r \cos\varphi$$

(on the surface of the cone). It remains to verify that

$$\sqrt{4R^2 + 4r^2 - 8rR\cos\varphi} + 2r\cos\varphi \geq 2R.$$

Since $2R - 2r\cos\varphi > 0$, it follows that by transporting $2r\cos\varphi$ to the right-hand side and squaring the new inequality we easily get the desired statement.

11.16. Let the angles between line l and lines OA, OB and OC be equal to α, β and γ. Then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

(Problem 1.21), and, therefore,

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2.$$

The sum of distances from points A, B and C to line l is equal to

$$a\sin\alpha + b\sin\beta + c\sin\gamma.$$

Let $x = \sin\alpha$, $y = \sin\beta$, $z = \sin\gamma$. In the problem we have to find the least and the greatest values of the quantity

$$ax + by + cz$$

provided

$$x^2 + y^2 + z^2 = 2, \quad 0 \leq x, y, z \leq 1.$$

These conditions single out a curvilinear triangle (Fig. 83) on the surface of the sphere

$$x^2 + y^2 + z^2 = 2.$$

Let the plane

$$ax + by + cz = p_0$$

be tangent to the surface of the sphere $x^2+y^2+z^2 = 2$ at point M_0 with coordinates (x_0, y_0, z_0), where $x_0, y_0, z_0 \geq 0$. Then

$$\begin{aligned} x_0 &= \lambda a, y_0 = \lambda b, z_0 = \lambda c, \lambda^2(a^2 + b^2 + c^2) = 2, \\ p_0 &= \lambda(a^2 + b^2 + c^2) = \sqrt{2(a^2 + b^2 + c^2)}. \end{aligned}$$

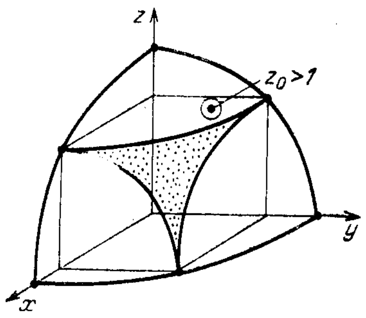

FIGURE 83 (SOL. 11.16)

If $z_0 \le 1$ (i.e., $c^2 \le a^2+b^2$), then M_0 belongs to the singled out curvilinear triangle and, therefore, in this case p_0 is the desired greatest value of the function $ax+by+cz$.

Now, let $z_0 > 1$, i.e., $c^2 > a^2 + b^2$. The plane $ax + by + cz = p$, where $p < p_0$ intersects the sphere under consideration along a circle. We are only interested in the values of p for which this circle intersects with the distinguished curvilinear triangle. The greatest of such p's corresponds to the value $z_0' = 1$. The problem to find x_0' and y_0' is, therefore, reduced to the problem: *for what x and y the quantity $ax + by$ takes the greatest value provided $x^2 + y^2 = 1$.*

It is easy to verify that $x_0' = \frac{a}{\sqrt{a^2+b^2}}$ and $y_0' = \frac{b}{\sqrt{a^2+b^2}}$, i.e., the greatest value of p is equal in this case to $\sqrt{a^2 + b^2 + c}$.

Now, let us prove that the least value of $ax + by + cz$ is attained on the distinguished triangle at vertex $x_1 = y_1 = 1, z_1 = 0$. Indeed, since $0 \le x, y, z \le 1$, then $x + y + z \ge x^2 + y^2 + z^2 = 2$ and, therefore, $y + z - 1 \ge 1 - x$. Both parts of this inequality are nonnegative and, therefore,

$$b(y + z - 1) \ge a(1 - x).$$

Hence,

$$ax + by + cz \ge ax + by + bz \ge a + b.$$

11.17. Let A be the intersection point of line l with the edge of the dihedral angle. On line l, draw a segment AB of length 1. Let B' be the projection of point B to the plane of another face and O be the projection of the point B to the edge of the dihedral angle. Then

$$\sin\angle BAB' = BB' = OB\sin\angle BOB' = \sin\angle BAO \sin\angle BOB'.$$

Since $\sin BOB'$ is the sine of the given dihedral angle, $\sin\angle BAB'$ takes its maximal value when $\angle BAO = 90°$.

11.18. Let $AA_1 = 1$, $AM = x$. Introduce a coordinate system whose axes are parallel to the prism's edges. The coordinates of vectors $\{MA_1\}$ and $\{MC_1\}$ are $(0, 1, -x)$ and $(2, 1, 2 - x)$; their inner product is equal to

$$1 - 2x + x^2 = (1 - x)^2 \ge 0.$$

Therefore, $\angle A_1MC_1 \le 90°$ and this angle is equal to $90°$ when $x = 1$.

11.19. There exists a parallelepiped $ABCDA_1B_1C_1D_1$ whose edges AA_1, BB_1 and CC_1 lie on the axes of the given cylinders (Problem 1.19); clearly, this parallelepiped is a cube with edge $2R$.

a) The distance from the center of this cube to either of the edges is equal to $\sqrt{2}R$ whereas the distance from any other point to at least one of the lines AA_1, DC and B_1C_1 is greater than $\sqrt{2}R$ (Problem 1.31). Therefore, the radius of the smallest ball tangent to all the three cylinders is equal to $(\sqrt{2}-1)R$.

b) Let K, L and M be the midpoints of edges AD, A_1B_1 and CC_1, i.e., the points where pairs of given cylinders are tangent. Then the triangle KLM is an equilateral one and its center O coincides with the center of the cube (Problem 1.3). Let K', L' and M' be the midpoints of edges B_1C_1, DC and AA_1; these points are symmetric to points K, L and M through O. Let us prove that the distance from line l that passes through point O perpendicularly to plane KLM to either of lines B_1C_1, DC and AA_1 is equal to $\sqrt{2}R$.

Indeed, $K'O \perp l$ and $K'O \perp B_1C_1$ and therefore, the distance between lines l and B_1C_1 is equal to $K'O = \sqrt{2}R$; for the other lines the proof is similar.

Therefore, the radius of the cylinder with axis l tangent to the three given cylinders is equal to $(\sqrt{2}-1)R$.

It remains to verify that the distance from any line l' that intersects triangle KLM to one of the points K', L', M' does not exceed $\sqrt{2}R$. Let, for example, the intersection point X of line l' with plane KLM lie inside triangle KOL. Then $M'X \leq \sqrt{2}R$.

11.20. In the process of the pulling the tetrahedron through the hole there will necessarily become a moment when vertex B is to one side of the hole's plane, vertex A is in the hole's plane and vertices C and D are to the other side of the hole's plane (or are in the hole's plane). At this moment let the plane of the hole intersect edges BC and BD at points M and N; then the hole's disk contains triangle AMN.

Now, let us find out for which positions of points M and N the radius of the disk that contains triangle AMN is the least possible.

First, suppose that triangle AMN is an acute one. Then the smallest disk that contains it is its circumscribed disk (cf. Problem l5.127). If the sphere whose equator is circumscribed about triangle AMN is not tangent to, say, edge BC, then inside this sphere on edge BC in a vicinity of point M we can select a point M' such that triangle $AM'N$ is still an acute one and the radius of its circumscribed circle is smaller than the radius of the circle circumscribed about triangle AMN. Therefore, in the position when the radius of the circle circumscribed about triangle AMN is minimal the considered sphere is tangent to edges BC and BD and, therefore, $BM = BN = x$.

Triangle AMN is an equilateral one and in it $MN = x$ and $AM = AN = \sqrt{x^2 - x + 1}$. Let K be the midpoint of MN, let L be the projection of B to plane AMN. Since the center of the sphere lies in this plane and lines BM and BN are tangent to the given sphere, we see that LN and LM are tangent to the circle circumscribed about triangle AMN. If $\angle MAN = \alpha$, then

$$LK = MK \tan\alpha = \frac{x^2\sqrt{3x^2 - 4x + 4}}{2(x^2 - 2x + 2)}.$$

In triangle AKB, angle $\angle AKB = \beta$ is an obtuse one and

$$\cos\beta = \frac{3x-2}{\sqrt{3(3x^2-4x+4)}}.$$

Therefore,

$$LK = -KB\cos\beta = \frac{x(2-3x)}{2\sqrt{3x^2-4x+4}}.$$

By equating the two expressions for LK we get an equation for x :

$$(1) \qquad 3x^3 - 6x^2 + 7x - 2 = 0.$$

The radius R of the circumscribed circle of triangle AMN is equal to $\frac{x^2-x+1}{\sqrt{3x^2-4x+4}}$. The approximate calculations for the root of the equation (the error not exceeding 0.00005) yield the values $x \approx 0.3913$, $R \approx 0.4478$.

Now, suppose that triangle AMN is not an acute one. Let $BM = x$, $BN = y$. Then

$$AM^2 = 1 - x + x^2,\ AN^2 = 1 - y + y^2 \text{ and } MN^2 = x^2 + y^2 - xy.$$

Angle $\angle MAN$ is an acute one because $AM^2 + AN^2 > MN^2$. Let, for definiteness, angle $\angle ANM$ be not acute, i.e.,

$$1 - x + x^2 \geq (x^2 + y^2 - xy) + (1 - y + y^2).$$

Then $0 \leq x \leq \frac{y(1-2y)}{1-y}$; hence, $y \leq 0.5$ and, therefore, $x \leq 2y(1-2y) \leq \frac{1}{4}$. On segment $[0, \frac{1}{2}]$, the quadratic $1 - x + x^2$ diminishes, hence,

$$AM^2 \geq 1 - \frac{1}{4} + \frac{1}{16} = \frac{13}{16} > (0.9)^2,$$

i.e., in the case of an acute triangle AMN the radius of the smallest disk that contains it is greater than for the case of an acute one.

Let us prove that the tetrahedron can pass through the hole of the found radius R. On the tetrahedron's edges draw segments of length x, where x is a root of equation (1), as indicated on Fig. 84 and perform the following sequence of motions:

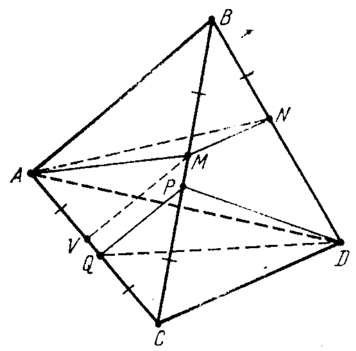

FIGURE 84 (SOL. 11.20)

a) let us place the tetrahedron so that the hole's circle becomes the circumscribed circle of triangle AMN and start rotating the tetrahedron about line MN until point V becomes in the hole's plane;

b) let us shift the tetrahedron so that plane VMN remains parallel to its initial position and points P and Q become on the hole's boundary;

c) let us rotate the tetrahedron about line PQ until vertex D becomes in the hole's plane.

Let us prove that all these operations are feasible. When we rotate the tetrahedron about line MN the hole's plane intersects it along the trapezoid whose diagonal diminishes from NA to NV and the acute angle at the greatest base increases to 90°. Therefore, the radius of the circle circumscribed about the trapezoid diminishes. Therefore, operation a) and, similarly, operation c) are feasible.

On edge BC, take point T. The section of tetrahedron $ABCD$ parallel to VMN and passing through point T is a rectangular with diagonal

$$\sqrt{t^2+(1-t)^2}=\sqrt{2(t-0.5)^2+0.5^2}, \text{ where } t=BT.$$

This implies the feasibility of operation b).

Answer: through an opening of radius 0.45 the tetrahedron can pass while it cannot pass through a hole of radius 0.44.

CHAPTER 12. CONSTRUCTIONS AND LOCI

§1. Skew lines

12.1. Find the locus of the midpoints of segments such that they are parallel to a given plane and their endpoints lie on two given skew lines.

12.2. Find the locus of the midpoints of segments of given length d whose endpoints lie on two given perpendicular skew lines.

12.3. Given three pairwise skew lines, find the locus of the intersection points of the medians of triangles parallel to a given plane and whose vertices lie on the given lines.

12.4. Given two skew lines in space and a point A on one of them. Through these given lines two perpendicular planes constituting a right dihedral angle are drawn. Find the locus of the projections of A on the edges of such dihedral angles.

12.5. Given line l and a point A. A line l' skew with l is drawn through A. Let MN be the common perpendicular to these two lines with point M on l'. Find the locus of such points M.

12.6. Pairwise skew lines l_1, l_2 and l_3 are perpendicular to one line and intersect it at points A_1, A_2 and A_3, respectively. Let M and N be points on lines l_1 and l_2, respectively, such that lines MN and l_3 intersect. Find the locus of the midpoints of segments MN.

12.7. Two perpendicular skew lines are given. The endpoints of segment A_1A_2 parallel to a given plane lie on the skew lines. Prove that all the spheres with diameters A_1A_2 have a common circle.

12.8. Points A and B move along two skew lines with constant but nonequal speeds; let k be the ratio of these speeds. Let M and N be points on line AB such that $AM : BM = AN : BN = k$ (point M lies on segment AB). Prove that points M and N move along two perpendicular lines.

§2. A sphere and a trihedral angle

12.9. Lines l_1 and l_2 are tangent to a sphere. Segment MN with its endpoints on these lines is tangent to the sphere at point X. Find the locus of such points X.

12.10. Points A and B lie on the same side with respect to plane Π so that line AB is not parallel to Π. Find the locus of the centers of spheres that pass through the given points and are tangent to the given plane.

12.11. The centers of two spheres of distinct radius lie in plane Π. Find the locus of points X in this plane through which one can draw a plane tangent to spheres: a) from the inside; b) from the outside. (We say that spheres are tangent from the *inside* if they lie on the different sides with respect to the tangent plane; they are tangent from the *outside* if the spheres lie on the same side with respect to the tangent plane).

* * *

12.12. Two planes parallel to a given plane Π intersect the edges of a trihedral angle at points A, B, C and A_1, B_1, C_1 respectively (we denote by the same letters points that lie on the same edge). Find the locus of the intersection points of planes ABC_1, AB_1C and A_1BC.

12.13. Find the locus of points the sum of whose distances from the planes of the faces of a given trihedral angle is a constant.

12.14. A circle of radius R is tangent to faces of a given trihedral angle all the planar angles of which are right ones. Find the locus of all the possible positions of its center.

§3. Various loci

12.15. In plane, an acute triangle ABC is given. Find the locus of projections to this plane of all the points X for which triangles ABX, BCX and CAX are acute ones.

12.16. In tetrahedron $ABCD$, height DP is the smallest one. Prove that point P belongs to the triangle whose sides pass through vertices of triangle ABC parallel to its opposite sides.

12.17. A cube is given. Vertices of a convex polyhedron lie on its edges so that on each edge exactly one vertex lies. Find the set of points that belong to all such polyhedrons.

12.18. Given plane quadrangle $ABCD$, find the locus of points M such that it is possible to intersect the lateral surface of pyramid $MABCD$ with a plane so that the section is a) a rectangle; b) a rhombus.

12.19. A broken line of length a starts at the origin and any plane parallel to a coordinate plane intersects the broken line not more than once. Find the locus of the endpoints of such broken lines.

§4. Constructions on plots

12.20. Consider cube $ABCDA_1B_1C_1D_1$ with fixed points P, Q, R on edges AA_1, BC, B_1C_1, respectively. Given a plot of the cubes's projection on a plane (Fig. 85). On this plot, construct the section of the cube with plane PQR.

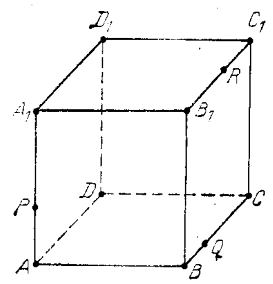

FIGURE 85

12.21. Consider cube $ABCDA_1B_1C_1D_1$ with fixed points P, Q, R on edges AA_1, BC and C_1D_1 respectively. Given a plot of the cubes's projection on a plane. On this plot, construct the section of the cube with plane PQR.

12.22. a) Consider trihedral angle $Oabc$ on whose faces Obc and Oac points A and B are fixed. Given the plot of its projection on a plane, construct the intersection point of line AB with plane Oab.

b) Consider a trihedral angle with three points fixed on its faces. Given a plot of its projection on a plane. On this plot, construct the section of the trihedral angle with the plane that passes through fixed points.

12.23. Consider a trihedral prism with parallel edges a, b and c on the lateral faces of which points A, B and C are fixed. Given the plot of its projection on a plane. On this plot, construct the section of the prism with plane ABC.

12.24. Let $ABCDA_1B_1C_1D_1$ be a convex hexahedron with tetrahedral faces. Given a plot of the three of the faces of this 6-hedron at vertex B (and, therefore, of seven of the vertices of the 6-hedron). Construct the plot of its 8-th vertex D_1.

§5. Constructions related to spatial figures

12.25. Given six segments in the plane equal to edges of tetrahedron $ABCD$, construct a segment equal to the height h_a of this tetrahedron.

12.26. Three angles equal to planar angles α, β and γ of a trihedral angle are drawn in the plane. Construct in the same plane an angle with measure equal to that of the dihedral angle opposite to the planar angle α.

12.27. Given a ball. In the plane, with the help of a compass and a ruler, construct a segment whose length is equal to the radius of this ball.

Solutions

12.1. Let given lines l_1 and l_2 intersect the given plane Π at points P and Q (if either $l_1 \parallel \Pi$ or $l_2 \parallel \Pi$, then there are no segments to be considered). Let us draw through the midpoint M of segment PQ lines l_1' and l_2' parallel to lines l_1 and l_2, respectively. Let a plane parallel to plane Π intersect lines l_1 and l_2 at points A_1 and A_2 and lines l_1' and l_2' at points M_1 and M_2, respectively. Then A_1A_2 is the desired segment and its midpoint coincides with the midpoint of segment M_1M_2 because $M_1A_1M_2A_2$ is a parallelogram. The midpoints of segments M_1M_2 lie on one line, since all these segments are parallel to each other.

12.2. The midpoint of any segment with the endpoints on two skew lines lies in the plane parallel to the skew lines and equidistant from them. Let the distance between the given lines be equal to a. Then the length of the projection to the considered "mid-plane" of a segment of length d with the endpoints on given lines is equal to $\sqrt{d^2 - a^2}$. Therefore, the locus to be found consists of the midpoints of segments of length $\sqrt{d^2 - a^2}$ with the endpoints on the projections of the given lines to the "mid-plane" (Fig. 86). It is easy to verify that $OC = \frac{AB}{2}$, i.e., the required locus is a circle with center O and radius $\frac{\sqrt{d^2-a^2}}{2}$.

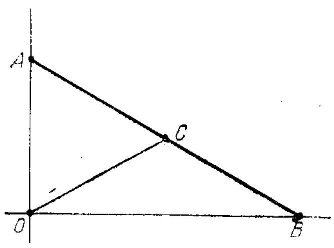

FIGURE 86 (SOL. 12.2)

12.3. The locus of the midpoints of sides AB of the indicated triangles is line l (cf. Problem 12.1). Consider the set of points that divide the segments parallel to the given plane with one endpoint on line l and the other one on the third of the given planes in ratio $1:2$. This set is the locus in question.

A slight modification in the solution of Problem 12.1 allows us to describe this locus further, namely to show that it is actually a line.

12.4. Let π_1 and π_2 be perpendicular planes passing through lines l_1 and l_2; let l be their intersection line; X the projection to l of point A that lies on line l_1. Let us draw plane Π through point A perpendicularly to l_2. Since $\Pi \perp l_2$, it follows that $\Pi \perp \pi_2$. Hence, line AX lies in plane Π and, therefore, if B is the intersection point of Π and l_2, then $\angle BXA = 90^\circ$, i.e., point X lies on the circle with diameter AB constructed in plane Π.

12.5. Let us draw the plane perpendicular to l through point A. Let M' and N' be the projections of points M and N to this plane. Since $MN \perp l$, it follows that $M'N' \parallel MN$. Line MN is perpendicular to plane AMM' because $NM \perp MM'$ and $NM \perp AM$. Hence, $NM \perp AM'$ and, therefore, point M' lies on the circle with diameter $N'A$. It follows that the locus to be found is a cylinder without two lines. The diametrically opposite generators of this cylinder are lines l and the line t that passes through point A parallel to l; the deleted lines are l and t.

12.6. The projection to a plane perpendicular to l_3 sends l_3 to point A_3; the projection $M'N'$ of line MN passes through this point; moreover, the projections of lines l_1 and l_2 are parallel. Therefore,

$$\{A_1M'\} : \{A_2N'\} = \{A_1A_3\} : \{A_2A_3\} = \lambda$$

is a constant, and, therefore, $\{A_1M\} = t\mathbf{a}$ and $\{A_2N\} = t\mathbf{b}$. Let O and X be the midpoints of segments A_1A_2 and MN. Then

$$2\{OX\} = \{A_1M\} + \{A_2N\} = t(\mathbf{a} + \mathbf{b}),$$

i.e., all the points X lie on one line.

12.7. Let B_1B_2 be the common perpendicular to given lines (points A_1 and B_1 lie on one given line). Since $A_2B_1 \perp A_1B_1$, point B_1 belongs to the sphere with diameter A_1A_2. Similarly, point B_2 lies on this sphere. The locus of the midpoints of segments A_1A_2, i.e., of the centers of the considered spheres is a line l (Problem 12.1). Any point of this line is equidistant from B_1 and B_2, hence, $l \perp B_1B_2$. Let M be the midpoint of segment B_1B_2;let O be the base of the perpendicular dropped to line l from point M. The circle of radius OB_1 with center O passing through points B_1 and B_2 is the one to be found.

12.8. Let A_1 and B_1 be positions of points A and B at another moment of time; Π a plane parallel to the given skew lines. Let us consider the projection to Π parallel to line A_1B_1. Let A', B', M' and N' be projections of points A, B, M and N, respectively; let C' be the projection of line A_1B_1. Points M and N move in fixed planes parallel to plane Π and, therefore, it suffices to verify that points M' and N' move along two perpendicular lines. Since

$$A'M' : M'B' = k = A'C' : C'B',$$

it follows that $C'M'$ is the bisector of angle $A'C'B'$. Similarly, $C'N'$ is the bisector of an angle adjacent to angle $A'C'B'$. The bisectors of two adjacent angles are perpendicular.

12.9. Let line l_1 that contains point M be tangent to the sphere at point A and line l_2 at point B. Let us draw through line l_1 the plane parallel to l_2 and consider the projection to this plane parallel to line AB. Let N' and X' be the images of points N and X under this projection. Since $AM = MX$ and $BN = NX$, we have

$$AM : AN' = AM : BN = XM : XN = X'M : X'N'$$

and, therefore, AX' is the bisector of angle MAN'. Hence, point X lies in the plane that passes through line AB and constitutes equal angles with lines l_1 and l_2 (there are two such planes). The desired locus consists of two circles without two points: the circles are those along which these planes intersect the given sphere and the points to be excluded are A and B.

12.10. Let C be the intersection point of line AB with the given plane, M the tangent point of one of the spheres to be found with plane Π. Since $CM^2 = CA \cdot CB$, it follows that point M lies on the circle of radius $\sqrt{CA \cdot CB}$ centered at C. Hence, the center O of the sphere lies on the lateral surface of a right cylinder whose base is this circle. Moreover, the center of the sphere lies in the plane that passes through the midpoint of segment AB perpendicularly to it.

Now, suppose that point O is equidistant from A and B and the distance from point C to the projection M of point O to plane Π is equal to $\sqrt{CA \cdot CB}$. Let CM_1 be the tangent to the sphere of radius OA centered at O. Then $CM = CM_1$ and, therefore,

$$OM^2 = CO^2 - CM^2 = CO^2 - CM_1^2 = OM_1^2,$$

i.e., point M belongs to the considered sphere. Since $OM \perp \Pi$, it follows that M is the tangent point of this sphere with plane Π.

Thus, the locus in question is the intersection of the lateral surface of the cylinder with the plane.

12.11. a) Let the given spheres intersect plane Π along circles S_1 and S_2. The common interior tangents to these circles split the plane into 4 parts. Let us consider the right circular cone whose axial section is the part that contains S_1 and S_2. The planes tangent to the given spheres from the inside are tangent to this cone. Any such plane intersects plane Π along the line that lies outside the axial section of the cone. The locus we are trying to find consists of points that lie outside the axial section of the cone (the boundary of the axial section belongs to the locus).

b) is solved similarly to heading a). We draw the common outer tangents and consider the axial section that consists of the part of the plane containing both circles and the part symmetric to it.

12.12. The intersection of planes ABC_1 and AB_1C is the line AM, where M is the intersection point of diagonals BC_1 and B_1C of trapezoid BCC_1B_1. Point M lies on line l that passes through the midpoints of segments BC and B_1C_1 and the vertex of the given trihedral angle (see Problem 1.22). Line l is uniquely determined by plane Π and, therefore, plane Π_a that contains line l and point A is also uniquely determined.

The intersection point of line AM with plane A_1BC belongs to plane Π_a because the whole line AM belongs to this plane. Let us construct plane Π_a similarly to Π_b. Let m be the intersection line of these planes (plane Π_c also passes through line m). The desired locus consists of points of this line that lie inside the given trihedral angle.

12.13. On the edges of the given trihedral angle with vertex O select points A, B and C the distance from which to the planes of faces is equal to the given number a. The area S of each of the triangles OAB, OBC and OCA is equal to $\frac{3V}{a}$, where V is the volume of tetrahedron $OABC$. Let point X lie inside trihedral angle $OABC$ and the distance from it to the planes of its faces be equal to a_1, a_2 and a_3. Then the sum of the volumes of the pyramids with vertex X and bases OAB, OBC and OCA is equal to $\frac{S(A_1+a_2+a_3)}{3}$. Therefore,

$$V = \frac{S(a_1 + a_2 + a_3)}{3} \pm v,$$

where v is the volume of tetrahedron $XABC$. Since $V = \frac{Sa}{3}$, it follows that $a_1 + a_2 + a_3 = a$ if and only if $v = 0$, i.e., X lies in plane ABC.

Let points A', B' and C' be symmetric to A, B and C, respectively, through point O. Since any point lies inside one of 8 trihedral angles formed by planes of the faces of the given trihedral angle, the locus in question is the surface of the convex polyhedron $ABCA'B'C'$.

12.14. Let us introduce a rectangular coordinate system directing its axes along the edges of the given trihedral angle. Let O_1 be the center of the circle; Π the plane of the circle, α, β and γ the angles between plane Π and coordinate planes. Since the distance from point O_1 to the intersection line of planes Π and Oyz is equal to R and the angle between these planes is equal to α, it follows that the distance from point O_1 to plane Oyz is equal to $R \sin \alpha$. Similar arguments show that the coordinates of point O_1 are

$$(R \sin \alpha, R \sin \beta, R \sin \gamma).$$

Since

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

(Problem 1.21), it follows that

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$$

and, therefore, $OO_1 = \sqrt{2}R$. Moreover, the distance from point O_1 to any face of the trihedral angle does not exceed R. The desired locus is a part of the sphere of radius $\sqrt{2}R$ centered at the origin and bounded by planes $x = R$, $y = R$ and $z = R$.

12.15. If angles XAB and XBA are acute ones, then point X lies between the planes drawn through points A and B perpendicularly to AB (for points X that do not lie on segment AB the converse is also true). Therefore, our locus lies inside (but not on the sides) of the convex hexagon whose sides pass through the vertices of triangle ABC perpendicularly to its sides (Fig. 87).

If the distance from point X to plane ABC is greater than the longest side of triangle ABC, then angles $\angle AXB$, $\angle AXC$ and $\angle BXC$ are acute ones. Therefore, the desired locus is the interior of the indicated hexagon.

12.16. It suffices to verify that the distance from point P to each side of triangle ABC does not exceed that from the opposite vertex. Let us prove this statement, for example, for side BC. To this end, let us consider the projection to the plane perpendicular to line BC; this projection sends points B and C to one point M

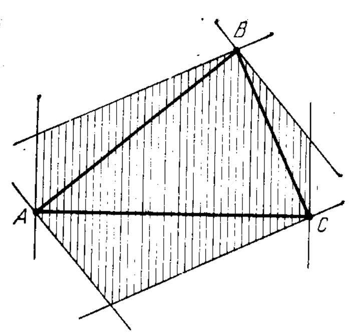

FIGURE 87 (SOL. 12.15)

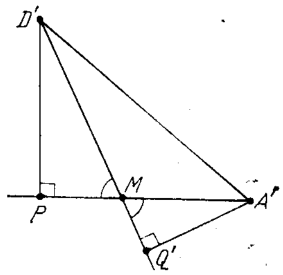

FIGURE 88 (SOL. 12.16)

(Fig. 88). Let $A'Q'$ be the projection of the corresponding height of the tetrahedron. Since $D'P \leq A'Q'$ by the hypothesis, $D'M \leq A'M$. It is also clear that $PM \leq D'M$.

12.17. Each of the considered polyhedrons is obtained from the given cube $ABCDA_1B_1C_1D_1$ by cutting off tetrahedrons from each of the trihedral angles at its vertices. The tetrahedron which is cut off the trihedral angle at vertex A is contained in tetrahedron AA_1BD. Thus, if we cut off the cube tetrahedrons, each of which is given by three edges of the cube that exit one point, then the remaining part of the cube is contained in any of the considered polyhedrons. It is easy to verify that the remaining part is an octahedron with vertices in the centers of the cube's faces. If the point does not belong to this octahedron, then it is not difficult to indicate a polyhedron to which it does not belong; for such a polyhedron we may take either tetrahedron AB_1CD_1 or tetrahedron A_1BC_1D.

12.18. Let P and Q be the intersection points of the extensions of the opposite sides of quadrilateral $ABCD$. Then MP and MQ are intersection lines of the planes of opposite faces of pyramid $MABCD$. The section of a pair of planes that intersect along line l is of the form of two parallel lines only if the pair of sections is parallel to l. Therefore, the section of pyramid $MABCD$ is a parallelogram only if the plane of the section is parallel to plane MPQ; the sides of the parallelogram are parallel to MP and MQ.

a) The section is a rectangular only if $\angle PMQ = 90°$, i.e., point M lies on the sphere with diameter PQ; the points of this sphere that lie in the plane of the given quadrilateral should be excluded.

b) Let K and L be the intersection points of the extensions of diagonals AC and BD with line PQ. Since the diagonals of the parallelogram obtained in the section of pyramid $MABCD$ are parallel to lines MK and ML, it follows that it is a rhombus only if $\angle KML = 90°$, i.e., point M lies on the sphere with diameter KL; the points of the sphere that lie in the plane of the given quadrilateral should be excluded.

12.19. Let (x, y, z) be coordinates of the endpoint of the broken line, (x_i, y_i, z_i) the coordinates of the vector of the i-th link of the broken line. The conditions of the problem imply that numbers x_i, y_i and z_i are nonzero and their sign is the same as that of numbers x, y and z, respectively. Therefore,

$$|x| + |y| + |z| = \sum(|x_i| + |y_i| + |z_i|)$$

and

$$|x_i| + |y_i| + |z_i| > l_i,$$

where l_i is the length of the i-th link of the broken line. Hence,

$$|x| + |y| + |z| > \sum l_i = a.$$

Moreover, the length of the vector (x, y, z) does not exceed the length of the broken line, i.e., it does not exceed a.

Now, let us prove that all the points of the ball of radius a centered at the origin lie outside the octahedron given by equation

$$|x| + |y| + |z| \le a$$

except for the points of coordinate planes that belong to the locus to be found. Let $M = (x, y, z)$ be a point on a face of the indicated octahedron. Then the broken line with vertices at points $(0, 0, 0)$, $(x, 0, 0)$, $(x, y, 0)$ and (x, y, z) is of length a. By "stretching" this broken line, i.e., by moving its endpoint along the ray OM we sweep over all the points of ray OM that lie between the sphere and the octahedron (except for the point on the octahedron's boundary).

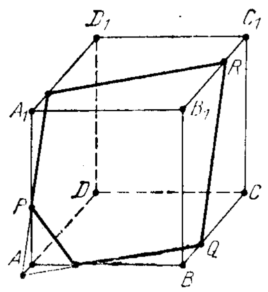

FIGURE 89 (SOL. 12.20)

12.20. In the process of construction we can make use of the fact that the lines along which a plane intersects a pair of parallel planes are parallel. The way of construction is clear from Fig. 89. First, we draw a line parallel to line RQ through point P and find the intersection points of this line with lines AD and A_1D_1. Then we connect these points with points Q and R and obtain sections of faces $ABCD$ and $A_1B_1C_1D_1$. On the section of one of the two remaining faces we have already constructed two points and now it only remains to connect them.

12.21. In this case the considerations used in the preceding problem are not sufficient for the construction. Therefore, let us first construct point M of intersection of line PR and the plane of face $ABCD$ as follows.

Point A is the projection of point P to the plane of face $ABCD$ and it is easy to construct the projection R' of point R to this plane (RC_1CR' is a parallelogram). Point M is the intersection point of lines PR and AR'. By connecting points M and Q we get the section of face $ABCD$. The further construction is performed by the same method as in the preceding problem (Fig. 90).

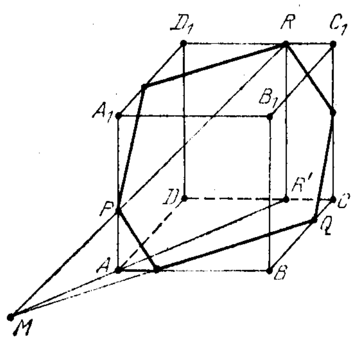

FIGURE 90 (SOL. 12.21)

12.22. a) Let P be an arbitrary point on edge c. Plane PAB intersects edges a and b at the same points at which lines PB and PA respectively intersect them. respectively. Denote these points by A_1 and B_1. Then the desired point is the intersection point of lines A_1B_1 and AB (Fig. 91).

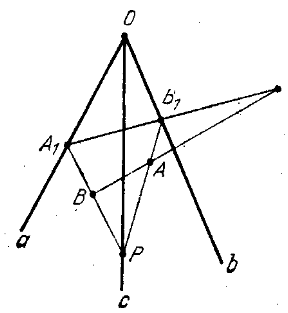

FIGURE 91 (SOL. 12.22)

b) Let points A, B and C be selected on faces Obc, Oac and Oab. By making use of part a) it is possible to construct the intersection point of line AB with plane OAb. Now, on plane Oab two points that belong to plane ABC are known: the just constructed point and point C. By connecting them we get the required section with plane Oab. The remaining part of the construction is obvious.

12.23. Let points A, B and C lie on the faces opposite to lines a, b and c. Let us construct intersection point X of line AB with the face in which point C lies. To this end let us select on line c an arbitrary point P and construct the section of the prism with plane PAB, i.e., let us find points A_1 and B_1 at which lines PA and PB intersect edges b and a, respectively. Clearly, X is the intersection point of lines AB and A_1B_1. Connecting points X and C we get the desired section of the face opposite to edge C. The remaining part of the construction is obvious.

12.24. First, let us construct the intersection line of planes of faces $ABCD$ and $A_1B_1C_1D_1$. The intersection point P of lines AB and A_1B_1 and the intersection point Q of lines BC and B_1C_1 belong to this plane. Let M be the intersection point of lines DA and PQ. Then M is the intersection point of face ADD_1A_1 with line PQ, i.e., point D_1 lies on line MA_1. Similarly, if N is the intersection point of lines CD and PQ, then point D_1 lies on line C_1N (Fig. 92).

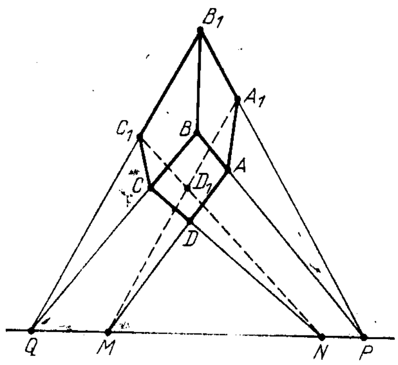

FIGURE 92 (SOL. 12.24)

12.25. Let us drop perpendicular AA_1 to plane BCD and perpendiculars AB', AC' and AD' to lines CD, BD and BC, respectively, from vertex A of tetrahedron $ABCD$. By the theorem on three perpendiculars $A_1B' \perp CD$, $A_1C' \perp BD$ and $A_1D' \perp BC$.

This implies the following construction. Let us construct the unfolding of tetrahedron $ABCD$ and drop heights from vertex A in all the faces that contain it (Fig. 93).

Point A_1 is the intersection point of these heights and the desired segment is a leg of a right triangle with hypothenuse AB' and a leg A_1B'.

12.26. Let us considered the trihedral angle with planar angles α, β and γ. Let O be its vertex. On the edge opposite to angle α, take point A and let us draw perpendiculars AB and AC to edge OA through point A in the planes of the faces. This construction can be performed on the given plane for the unfolding of the trihedral angle (Fig. 94). Let us now construct triangle $BA'C$ with sides $BA' = BA_1$ and $CA' = CA_2$. Angle $BA'C$ is the one to be constructed.

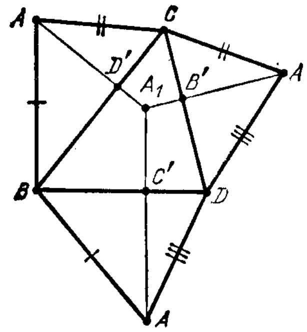

FIGURE 93 (SOL. 12.25)

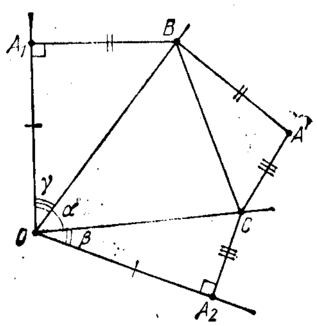

FIGURE 94 (SOL. 12.26)

12.27. On the given ball, let us construct with the help of a compass a circle with center A and, on this circle, fix three distinct arbitrary points. With the help of a compass it is easy to construct on a plane a triangle equal to the triangle with vertices at these points. Next, let us construct the circle circumscribed about this triangle and consequently find its radius.

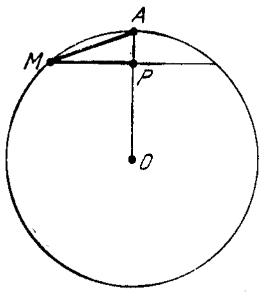

FIGURE 95 (SOL. 12.27)

Let us consider the section of the given ball that passes through its center O, point A and a point M of the circle constructed on the ball. Let P be the base of the perpendicular dropped from M to segment OA (Fig. 95). The lengths of segments AM and MP are known and, therefore, it is possible to construct segment AO.

CHAPTER 13. CERTAIN PARTICULAR METHODS FOR SOLVING PROBLEMS

§1. The principle of extremal element

13.1. Prove that every tetrahedron contains an edge that forms acute angles with the edges that go out of its endpoints.

13.2. Prove that in every tetrahedron there is a trihedral angle at a vertex with all the plane angles being acute ones.

13.3. Prove that in any tetrahedron there are three edges that go out of one vertex such that from these edges a triangle can be constructed.

13.4. A regular n-gon $A_1 \dots A_n$ lies at the base of pyramid $A_1 \dots A_nS$. Prove that if

$$\angle SA_1A_2 = \angle SA_2A_3 = \dots = \angle SA_nA_1,$$

then the pyramid is a regular one.

13.5. Given a right triangular prism $ABCA_1B_1C_1$, find all the points on face ABC equidistant from lines AB_1, BC_1 and CA_1.

13.6. On each of $2k+1$ planets sits an astronomer who observes the planet nearest to him (all the distances between planets are distinct). Prove that there is a planet that nobody observes.

13.7. There are several planets — unit spheres — in space. Let us fix on each planet the set of all the points from which none of the other planets is seen. Prove that the sum of the areas of the fixed parts is equal to the surface area of one of the planets.

13.8. Prove that the cube cannot be divided into several distinct small cubes.

§2. Dirichlet's principle

13.9. Prove that any convex polyhedron has two faces with an equal number of sides.

13.10. Inside a sphere of radius 3 several balls the sum of whose radii is equal to 25 are placed (these balls can intersect). Prove that for any plane there exists a plane parallel to it and intersecting at least 9 inner balls.

13.11. A convex polyhedron P_1 with nine vertices $A_1, A_2, \dots, A_9$ is given. Let $P_2, P_3, \dots, P_9$ be polyhedrons obtained from the given one by parallel translations by vectors $\{A_1A_2\}, \dots, \{A_1A_9\}$, respectively. Prove that at least two of 9 polyhedrons $P_1, P_2, \dots, P_9$ have a common interior point.

13.12. A searchlight that lights a right trihedral angle (*octant*) is placed in the center of a cube. Is it possible to turn it so that it doesn't light any of the cube's vertices?

13.13. Given a regular tetrahedron with edges of unit length, prove the following statements:

a) on the surface of the tetrahedron 4 points can be fixed so that the distance from any point on the surface to one of these four points would not exceed 0.5;

b) it is impossible to fix 3 points on the surface of the tetrahedron with the above property.

§3. Entering the space

While solving planimetric problems the consideration that the plane can be viewed as lying in space and, therefore, some auxiliary elements outside the given plane can be used is sometimes of essential help. Such a method for solving planimetric problems is called *entering the space* method.

13.14. Along 4 roads each of the form of a straight line no two of which are parallel and no three of which pass through one point, 4 pedestrians move with constant speeds. It is known that the first pedestrian met the second one, third one and fourth one, and the second pedestrian met the third and the fourth ones. Prove that then the third pedestrian met the fourth one.

13.15. Three lines intersect at point O. Points A_1 and A_2 are taken on the first line, points B_1 and B_2 are taken on the second line, points C_1 and C_2 are taken on the third one. Prove that the intersection points of lines A_1B_1 and A_2B_2, B_1C_1 and B_2C_2, A_1C_1 and A_2C_2 lie on one line (we assume that the lines intersect, i.e., are not parallel).

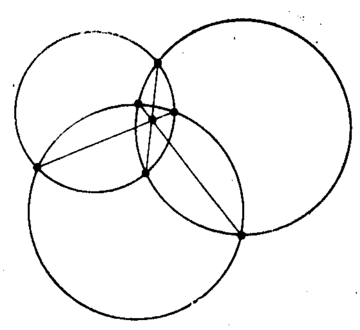

FIGURE 95 (13.16)

13.16. Three circles intersect pairwise and are placed as plotted on Fig. 96. Prove that the common chords of the pairs of these circles intersect at one point.

13.17. Common exterior tangents to three circles on the plane intersect at points A, B and C. Prove that these points lie on one line.

13.18. What least number of bands of width 1 are needed to cover a disk of diameter d?

13.19. On sides BC and CD of square $ABCD$, points M and N are taken such that $CM+CN=AB$. Lines AM and AN divide diagonal BD into three segments. Prove that from these segments one can always form a triangle one angle of which is equal to 60°.

13.20. On the extensions of the diagonals of a regular hexagon, points K, L and M are fixed so that the sides of the hexagon intersect the sides of triangle KLM at six points that are vertices of a hexagon H. Let us extend the sides of hexagon H that do not lie on the sides of triangle KLM. Let P, Q, R be their intersection points. Prove that points P, Q, R lie on the extensions of the diagonals of the initial hexagon.

13.21. Consider a lamina analogous to that plotted on Fig. 97 a) but composed of $3n^2$ rhombuses. It is allowed to interchange rhombuses as shown on Fig. 98.

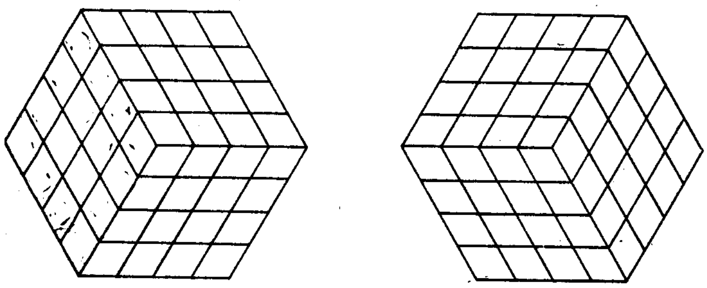

FIGURE 97 13.21)

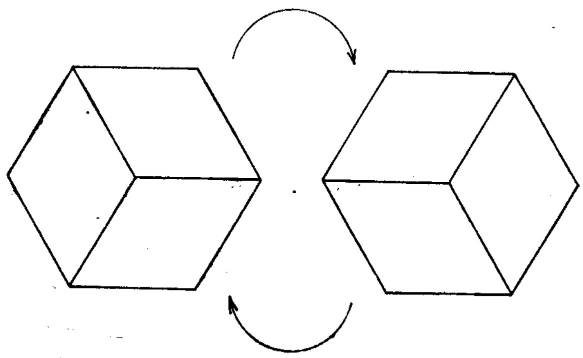

FIGURE 98 (13.21)

What is the least possible number of such operations required to get the lamina plotted on Fig. 97 b)?

13.22. A regular hexagon is divided into parallelograms of equal area. Prove that the number of the parallelograms is divisible by 3.

13.23. Quadrilateral $ABCD$ is circumscribed about a circle and its sides AB, BC, CD and DA are tangent to the circle at points K, L, M and N, respectively. Prove that lines KL, MN and AC either intersect at one point or are parallel.

13.24. Prove that the lines intersecting the opposite vertices of a circumscribed hexagon intersect at one point. (Brianchon's theorem.)

13.25. A finite collection of points in plane is given. A *triangulation* of the plane is a set of nonintersecting segments with the endpoints at the given points such that any other segment with endpoints at the given points intersects at least one of the given segments (Fig. 99). Prove that there exists a triangulation such that none of the circumscribed circles of the obtained triangles contains inside it any other of the given points and if no 4 of the given points lie on one circle, then such a triangulation is unique.

* * *

13.26. On the plane three rays with a common source are given and inside each of the angles formed by these rays a point is fixed. Construct a triangle so that its vertices would lie on the given rays and sides would pass through the given points.

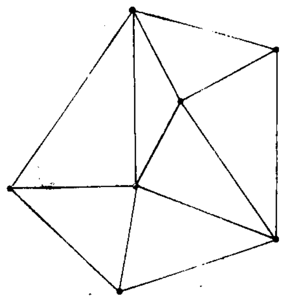

FIGURE 99 (13.25)

13.27. Given three parallel lines and three points on the plane. Construct a triangle whose sides (or their extensions) pass through the given points and whose vertices lie on the given lines.

Solutions

13.1. If AB is the longest side of triangle ABC, then $\angle C \geq \angle A$ and $\angle C \geq \angle B$; therefore, both angles A and B should be acute ones. Thus, all the acute angles are adjacent to the longest edge of the tetrahedron.

13.2. The sum of the angles of each face is equal to π and any tetrahedron has 4 faces. Therefore, the sum of all the plane angles of a tetrahedron is equal to 4π. Since a tetrahedron has 4 vertices, there exists a vertex the sum of whose planar angles does not exceed π. Hence, all the plane angles at this vertex are acute ones because any plane angle of a trihedral angle is smaller than the sum of the other two planar angles (Problem 5.4).

13.3. Let AB be the longest edge of tetrahedron $ABCD$. Since

$$(AC+AD-AB)+(BC+BD-BA)=(AD+BD-AB)+(AC+BC-AB)\geq 0,$$

it follows that either

$$AC+AD-AB>0$$

or

$$BC+BD-BA>0.$$

In the first case the triangle can be formed of the edges that exit vertex A and in the second one of the edges that exit vertex B.

13.4. On the plane, let us construct angle $\angle BAC$ equal to α, where $\alpha = \angle SA_1A_2 = \dots = \angle SA_nA_1$. Let us assume that the length of segment AB is equal to that of the side of the regular polygon serving as the base of the pyramid. Then for each $i = 1, \dots, n$ one can construct point S_i on ray AC so that $\triangle AS_iB = \triangle A_iSA_{i+1}$.

Suppose not all points S_i coincide. Let S_k be the point nearest to B and S_l the point most distant from B. Since $S_kS_l > |S_kB - S_lB|$, we have $|S_kA - S_lA| > |S_kB - S_lB|$, i.e., $|S_{k-1}B - S_{l-1}B| > |S_kB - S_lB|$. But in the right-hand side of the latter inequality there stands the difference between the greatest and the smallest numbers and in the left-hand side the difference of two numbers confined between these two extreme ones. Contradiction. Hence, all the points S_i coincide and, therefore, point S is equidistant from the vertices of base $A_1 \dots A_n$.

13.5. Let O be the point on face ABC equidistant from the mentioned lines. We may assume that A is the most distant from O point of base ABC. Let us consider triangles AOB_1 and BOC_1. Sides AB_1 and BC_1 of these triangles are equal and these are the longest sides (cf. Problem 10.5), i.e., the bases of the heights dropped to these sides lie on the sides themselves. Since these heights are equal, the inequality $AO \geq BO$ implies $OB_1 \leq OC_1$. In right triangles $\angle BB_1O$ and $\angle CC_1O$ legs BB_1 and CC_1 are equal and, therefore, $BO \leq CO$.

Thus, the inequality $AO \geq BO$ implies $BO \leq CO$. By similar argument we deduce that $CO \geq AO$ and $AO \leq BO$. Therefore, $AO = BO = CO$, i.e., O is the center of equilateral triangle ABC.

13.6. Let us consider a pair of planets, A and B, with the shortest distance between them. Then the astronomers observe the each other's planets: the astronomer of planet A observes planet B and the astronomer from planet B observes planet A. The following two cases are possible:

1) At least one of the planets, A or B, is observed by some other astronomer. Then for $2k-1$ planets there remain $2k-2$ observers and, therefore, there is a planet which nobody observes.

2) None of the remaining astronomers observes either planet A or planet B. Then this pair of planets can be discarded; let us consider a similar system with the number of planets smaller by 2. In the end either we either encounter the first situation or there remains one planet which nobody observes.

13.7. First, let us consider the case of two planets. Each of them is divided by the equator perpendicular to the segment that connects the centers of the planets into two hemispheres such that from one hemisphere the other planet is seen and from the other one it is not seen.

Notice that in order to be meticulous one should have to be more precise in the formulation of the problem: how one should treat the points of these equators, should one think that the other planet is seen from them or not? But since the area of both equators is equal to zero this is actually immaterial. Therefore, in what follows we will disregard the equatorial points.

Let $O_1, \dots, O_n$ be the centers of the given planets. It suffices to prove that for any vector $\mathbf{a}$ of length 1 there exists a point X on the i-th planet for which $\{O_iX\} = \mathbf{a}$ and no other planet is seen from X; such a point is unique.

First, let us prove the uniqueness of point X. Suppose that $\{O_iX\} = \{O_jY\}$ and no other planet is seen from either X or Y. But from the considered above case of two planets it follows that if the j-th planet is not seen from point X, then the i-th planet will be seen from point Y. Contradiction.

Now, let us prove the existence of point X. Introduce a coordinate system directing Ox-axis along vector $\mathbf{a}$. Then the point on given planets for which the coordinate x takes the greatest value is the desired one.

13.8. Suppose that the cube is divided into several distinct small cubes. Then each of the faces of the cube becomes divided into small squares. Let us select the smallest of all the squares on each face. It is not difficult to see that the smallest of the small squares of the division of a square — a face — cannot be adjacent to its boundary. Therefore, the small cube whose base is the selected smallest small square lies inside the "well" formed by the cubes adjacent to its lateral faces. Thus, its face opposite to the base should be filled in by yet smaller small cubes. Let us select the smallest among them and repeat for it the same arguments.

By continuing in this way we finally reach the opposite face and discover on it a small square of the partition which is smaller than the one with which we have started. But we have started with the smallest of all the small squares of the partitions of the cube's faces. Contradiction.

13.9. Let the number of the faces of the polyhedron be equal to n. Then each of its faces can have 3 to $n-1$ sides, i.e., the number of sides on each of its n faces can take one of $n-3$ values. Therefore, there are 2 faces with an equal number of sides.

13.10. Let us consider the projection to a line perpendicular to the given plane. This projection sends the given ball to a segment of length 3 and the inner balls to segments the sum of whose lengths is equal to 25. Suppose that the sought for plane does not exist, i.e., any plane parallel to the given one intersects not more than 8 of the inner balls. Then any point on the segment of length 3 belongs to not more than 8 segments — the projections of the inner balls. It follows that the sum of the lengths of these segments does not exceed 24. Contradiction.

13.11. Let us consider the polyhedron P which is the image of polyhedron P_1 under the homothety with center A_1 and coefficient 2. Let us prove that all 9 polyhedrons lie inside P. Let $A_1, A_2^*, \dots, A_9^*$ be the vertices of P. Let us prove that, for instance, polyhedron P_2 lies inside P. To this end it suffices to notice that the parallel translation by vector $\{A_1A_2\}$ sends points $A_1, A_2, A_3, \dots, A_9$ into points $A_2, A_2^*, A_3', \dots, A_9'$, respectively, where A_i' is the midpoint of segment $A_2^*A_i^*$.

The sum of volumes of polyhedrons $P_1, P_2, \dots, P_9$ that lie inside polyhedron P is equal to $9V$, where V is the volume of P_1, and the volume of P is equal to $8V$. Therefore, the indicated 9 polyhedrons cannot help having common inner points.

13.12. First, let us prove that it is possible to rotate the searchlight so that it would light neighbouring vertices of the cube, say A and B. If $\angle AOB < 90^\circ$, then from the center O of the cube we can light segment AB. To this end it suffices to place segment AB in one of the faces that the seasrchlight lights and then slightly move the seasrchlight. It remains to verify that $\angle AOB < 90^\circ$. This follows from the fact that

$$AO^2 + BO^2 = \frac{3}{4}AB^2 + \frac{3}{4}AB^2 > AB^2.$$

Let us move the searchlight so that it would light two vertices of the cube. The planes of faces of the angle lighted by the searchlight divide the space into 8 octants. Since two of eight vertices of the cube lie in one of these octants, there exists an octant which does not contain any vertex of the cube. This octant determines the required position of the sesarchlight.

Remark. We did not consider the case when one of the planes of octant's faces contains a vertex of the cube. This case can be get rid of by slightly moving the searchlight.

13.13. a) It is easy to verify that the midpoints of edges AB, BC, CD, DA have the desired property. Indeed, two edges of each of the faces have fixed points. Now, let us consider, for example, face ABC. Let B_1 be the midpoint of edge AC. Then triangles ABB_1 and CBB_1 are covered by disks of radius 0.5 with the centers at the midpoints of sides AB and CD, respectively.

b) On the surface of the tetrahedron fix three points and consider the part of the surface of the tetrahedron covered by balls of radius 0.5 centerd at these points.

We will say that an angle of the face is *covered* if for some number $\epsilon > 0$ all the points of the face distant from the vertex of the given angle not further than ϵ are covered. It suffices to prove that for the case of three points a non-covered angle of the face always exists.

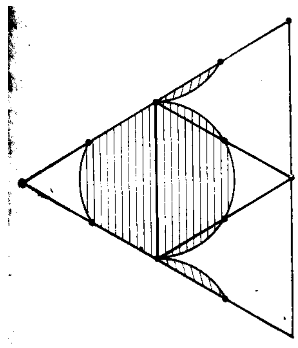

FIGURE 100 (SOL. 13.13)

If the ball of radius 0.5 centered at O covers two points, A and B, the distance between which is equal to 1, then O is the midpoint of segment AB. Therefore, if a ball of radius 0.5 covers two vertices of the tetrahedron then its center is the midpoint of the edge that connects these vertices.

It is clear from Fig. 100 that in this case the ball covers 4 angles of the faces. For the uncovered angles their bisectors are also uncovered and therefore, it cannot happen that every single ball does not cover an angle but all the balls together do cover it. It is also clear that if a ball only covers one vertex of the tetrahedron then it only covers three angles.

There are 12 angles of the faces in the tetrahedron altogether. Therefore, 3 balls of radius 0.5 each can cover them only if the centers of the balls are the midpoints of the tetrahedron's edges and not of arbitrary edges but of non-adjacent edges because the balls with centers in the midpoints of adjacent edges have a common angle covered by them. Clearly, it is impossible to select three pairwise nonadjacent edges in a tetrahedron.

13.14. In addition to the coordinates in plane in which the pedestrians move introduce the third coordinate system, the axis of time. Then consider the graphs of the pedestrians' movements. Clearly, the pedestrians meet when the graphs of their movements intersect. As follows from the hypothesis, the graphs of the third and the fouth pedestrians lie in the plane determined by the graphs of the first two pedestrians (Fig. 101). Therefore, the graphs of the third and the fourth pedestrians intersect.

13.15. In space, let us take points C_1' and C_2' so that their projections are C_1 and C_2 and the points themselves do not lie in the initial plane. Then the projections of the intersection points of lines A_1C_1' and A_2C_2', B_1C_1' and B_2C_2' are the intersection points of lines A_1C_1 and A_2C_2, B_1C_1 and B_2C_2, respectively. Therefore, the points indicated in the formulation of the problem lie on the projection of the intersection line of planes $A_1B_1C_1'$ and $A_2B_2C_2'$, where line $C_1'C_2'$ contains point O.

13.16. Let us construct spheres for which our circles are equatorial circles. Then the common chords of pairs of these circles are the projections of the circles along

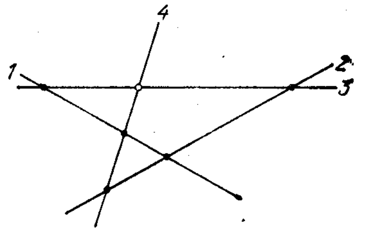

FIGURE 101 (SOL. 13.14)

which the constructed spheres intersect. Therefore, it suffices to prove that the spheres have a common point. To this end let us consider a circle along which the two of our spheres intersect. One endpoint of the diameter of this circle that lies in the initial plane is outside the third sphere whereas its other endpoint is inside it. Therefore, the circle intersects the sphere, i.e., the three spheres have a common point.

13.17. For each of our circles consider the cone whose base is the given circle and height is equal to the radius of the circle. Let us assume that these cones are situated to one side of the initial plane. Let O_1, O_2, O_3 be the centers of the circles and O_1', O_2', O_3' the vertices of the corresponding cones. Then the intersection point of common exterior tangents to the i-th and j-th circles coincides with the intersection point of line $O_i'O_j'$ with the initial plane. Thus, points A, B and C lie on the intersection line of plane $O_1'O_2'O_3'$ with the initial plane.

13.18. In the solution of this problem let us make use of the face that the area of the ribbon cut on the sphere of diameter d by two parallel planes the distance between which is equal to h is equal to πdh (see Problem 4.24).

Let a disk of diameter d be covered by k ribbons of width 1 each. Let us consider the sphere for which this disk is the equatorial one. By drawing planes perpendicular to the equator through the boundaries of the ribbons we get spherical ribbons on the sphere such that the area of each of the ribbons is equal to πd (more precisely, does not exceed πd because one of the boundaries of the initial ribbon might not intersect the disk). These spherical ribbons also cover the whole sphere and, therefore, their area is not less than the area of the sphere, i.e., $k\pi d \geq \pi d^2$ and $k \geq d$. Clearly, if $k \geq d$, then k ribbons can cover the disk of diameter d.

13.19. Let us complement square $ABCD$ to cube $ABCDA_1B_1C_1D_1$. The hypothesis of the problem implies that $CM = DN$ and $BM = CN$. On edge BB_1, fix point K so that $BK = DN$. Let segments AM and AN intersect diagonal BD at points P and Q, let R be the intersection point of segments AK and BA_1. Let us prove that sides of triangle PBR are equal to the corresponding segments of diagonal BD. It is clear that $BR = DQ$. Now, let us prove that $PR = PQ$. Since $BK = CM$ and $BM = CN$, it follows that $KM = MN$ and, therefore, $\triangle AKM = \triangle ANM$. Moreover, $KR = NQ$; hence, $RP = PQ$. It remains to notice that $\angle RBP = \angle A_1BD = 60°$ because triangle A_1BD is an equilateral one.

13.20. Let us denote the initial hexagon by $ABCC_1D_1A_1$ and let us assume that it is the projection of cube $A'B'C'D'A_1'B_1'C_1'D_1'$ on the plane perpendicular to diagonal $D'B_1'$. Let K', L', M' be points on lines $B_1'C_1'$, $B_1'B'$ and $B_1'A_1'$ whose projections are K, L and M, respectively (Fig. 102).

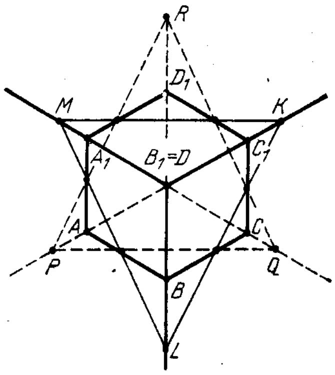

FIGURE 102 (SOL. 13.20)

Then H is the section of the cube by plane $K'L'M'$, in particular, the sides of triangle PQR lie on the projections of the lines along which plane $K'L'M'$ intersects the planes of the lower faces of the cube (we assume that point B_1' lies above point D'). Hence, points P, Q, R are the projections of the intersection points of the extensions of the lower edges of the cube ($D'A'$, $D'C'$, $D'D_1'$) with plane $K'L'M'$, and, therefore, they lie on the extensions of the diagonals of the initial hexagon.

13.21. Let us consider the projection of the cube composed of n^3 smaller cubes to the plane perpendicular to its diagonal. Then we can consider Fig. 97 a) as the projection of the whole of this cube and Fig. 97 b) as the projection of the back faces of the cube only.

The admissible operation is the insertion or removal of the cube provided one inserts the cube so that some three of its faces only touch the already existing faces. It is clear that it is impossible to remove n^3 small cubes for fewer than n^3 operations whereas it is possible to do so in n^3 operations.

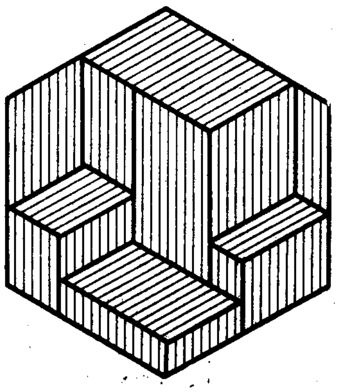

FIGURE 103 (SOL. 13.22)

13.22. A regular hexagon divided into parallelograms can be represented as the projection of a cube from which several rectangular parallelepipeds are cut off (Fig. 103). Then the projections of the rectangles parallel to the cube's faces cover the faces in one coat. Therefore, in the initial hexagon the sum of the areas of the parallelograms of each of the three types (parallelograms of one type have parallel sides) is equal to $\frac{1}{3}$ of the area of the hexagon. Since the parallelograms are of equal area, the number of parallelograms of each type should be the same. Therefore, their total number is divisible by 3.

13.23. Let us draw perpendiculars through the vertices of quadrilateral $ABCD$ to the plane in which it lies. On the the perpendiculars let us draw segments AA', BB', CC' and DD' equal to the tangents drawn to the circle from the corresponding vertices of the quadrilateral so that points A' and C' lie on the same side with respect to the given plane and B' and D' lie on the other side (Fig. 104). Since $AA' \parallel BB'$ and $\angle AKA' = 45° = \angle BKB'$, point K lies on segment $A'B'$. Similarly, point L lies on segment $B'C'$ and, therefore, line KL lies in plane $A'B'C'$. Similarly, line MN lies in plane $A'D'C'$.

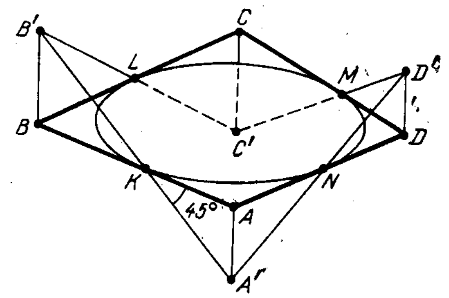

FIGURE 104 (SOL. 13.23)

If line $A'C'$ is parallel to the initial plane, then lines AC, KL and MN are parallel to line $A'C'$. Now, let line $A'C'$ intersect the initial plane at point P, i.e., let P be the intersection point of planes $A'B'C'$, $A'D'C'$ and the initial plane. Then lines KL, AC and MN pass through point P.

13.24. Let us draw perpendiculars through vertices of the hexagon $ABCDEF$ to the plane in which it lies and draw segments AA', ..., FF' on them equal to the tangents drawn to the circles from the corresponding vertices; let this be drawn so that points A', C' and E' lie to one side of the given plane and B', D' and F' lie to the other side (Fig. 105). Let us prove that lines $A'B'$ and $E'D'$ lie in one plane. If $AB \parallel ED$, then $A'B' \parallel E'D'$. If lines AB and ED intersect at point P, then let us draw on the perpendicular to the initial plane through point P segments PP' and PP'' equal to the tangent to the circle drawn from point P.

Let Q be the tangent point of the circle with side AB. Then segments $P'Q$, $P''Q$, $A'Q$ and $B'Q$ form angles of 45° with line AB and lie in the plane perpendicular to the given plane and passing through line AB. Therefore, line $A'B'$ passes through either point P' or P''. It is not difficult to verify that line $E'D'$ also passes through the same point. Therefore, lines $A'B'$ and $E'D'$ intersect, hence, lines $A'D'$ and $B'E'$ also intersect.

We similarly prove that lines $A'D'$, $B'E'$ and $C'F'$ intersect pairwise. But since these lines do not lie in one plane, they should intersect at one point. Lines AD, BE and CF pass through the projectioin of this point to the given plane.

13.25. Let us take an arbitrary sphere tangent to the given plane and consider the stereographic projection of the plane to the sphere. We get a finite set of points

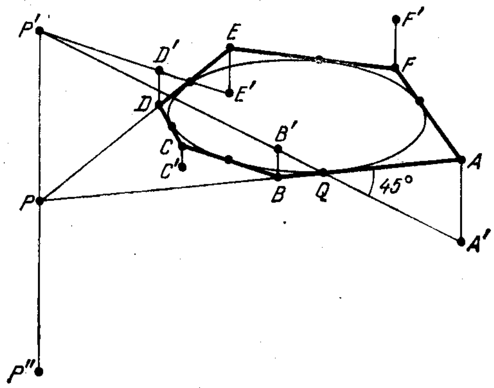

FIGURE 105 (SOL. 13.24)

on the sphere which are vertices of a convex polyhedron. To get the desired triangulation, we have to connect those of the given points whose images on the sphere are connected by the edges of the obtained convex polyhedron. The uniqueness of the triangulation is equivalent to the fact that all the faces of the polyhedron are triangles which, in turn, is equivalent to the fact that no four of the given points lie on one circle.

13.26. It is possible to represent the given rays and points as a plot of the projection of a trihedral angle with three points fixed on its faces. The problem requires to construct a section of this angle with the plane that passes through the given points. The corresponding construction is described in the solution of Problem 12.22 b).

13.27. It is possible to represent the given lines as the projections of lines on which the edges of the trihedral prism lie and the given points as the projections of points that lie on the faces (or their extensions) of this prism. The problem requires to construct the section of the prism with the plane that passes through the given points. The corresponding construction is described in the solution of Problem 12.23.

CHAPTER 14. THE CENTER OF MASS. THE MOMENT OF INERTIA. BARYCENTRIC COORDINATES

§1. The center of mass and its main properties

Let there be given a system of mass points in space, i.e., a set of pairs (X_i, m_i), where X_i is a point in space and m_i is a number such that $m_1 + \dots + m_n \neq 0$. The *center of mass* of the system of points $X_1, \dots, X_n$ with masses $m_1, \dots, m_n$ respectively is a point O such that $m_1\{OX_1\} + \dots + m_n\{OX_n\} = \{0\}$.

14.1. a) Prove that the center of mass of any (finite) system of points exists and is unique.

b) Prove that if X is an arbitrary point on the plane and O is the center of mass of points $X_1, \dots, X_n$ whose masses are equal to $m_1, \dots, m_n$, respectively, then

$$\{XO\} = \frac{1}{m_1 + \dots + m_n}(m_1\{XX_1\} + \dots + m_n\{XX_n\}).$$

14.2. Prove that the center of mass of a system of points $X_1, \dots, X_n$; $Y_1, \dots, Y_m$ whose masses are equal to $a_1, \dots, a_n$; $b_1, \dots, b_m$, respectively, coincides with the center of mass of two points: the center of mass X of the first system with mass $a_1 + \dots + a_n$ and the center of mass Y of the other system with mass $b_1 + \dots + b_m$.

14.3. a) Prove that the segments that connect the vertices of a tetrahedron with the intersection points of the medians of the opposite faces intersect at one point and each of them is divided at this point at the ratio 3:1 counting from the vertex. (These segments are called the *medians of the tetrahedron.*)

b) Prove that the segments that connect the midpoints of the opposite edges of the tetrahedron also intersect at the same point and each of them is divided by this point in halves.

14.4. Given parallelepiped $ABCDA_1B_1C_1D_1$ and plane A_1DB that intersects diagonal AC_1 at point M, prove that $AM : AC_1 = 1 : 3$.

14.5. Given triangle ABC and line l; let A_1, B_1 and C_1 be arbitrary points on l. Find the locus of the centers of mass of triangles with vertices in the midpoints of segments AA_1, BB_1 and CC_1.

14.6. On edges AB, BC, CD and DA of tetrahedron $ABCD$ points K, L, M and N, respectively, are taken so that $AK : KB = DM : MC = p$ and $BL : LC = AN : ND = q$. Prove that segments KM and LN intersect at one point, O, such that $KO : OM = q$ and $NO : OL = p$.

14.7. On the extensions of the heights of tetrahedron $ABCD$ beyond the vertices segments AA_1, BB_1, CC_1 and DD_1 whose lengths are inverse proportional to the heights are depicted. Prove that the centers of mass of tetrahedrons $ABCD$ and $A_1B_1C_1D_1$ coincide.

14.8. Two planes intersect the lateral edges of a regular n-gonal prism at points $A_1, \dots, A_n$ and $B_1, \dots, B_n$, respectively, and these planes do not have common points inside the prism. Let M and N be the centers of mass of polygons $A_1 \dots A_n$ and $B_1 \dots B_n$.

a) Prove that the sum of lengths of segments $A_1B_1, \dots, A_nB_n$ is equal to nMN.

b) Prove that the volume of the part of the prism confined between these planes is equal to sMN, where s is the area of the base of the prism.

§2. The moment of inertia

The quantity $I_M = m_1MX_1^2 + \dots + m_nMX_n^2$ is called the *moment of inertia* relative point M of the system of points $X_1, \dots, X_n$ with masses $m_1, \dots, m_n$ respectively.

14.9. Let O be the center of mass of a system of points whose total mass is equal to m. Prove that the moments of inertia of this system relative point O and relative an arbitrary point X are related by the formula

$$I_X = I_O + m \times XO^2.$$

14.10. a) Prove that the moment of inertia with respect to the center of mass of a system of points of unit mass each is equal to $\frac{1}{n}\sum_{i<j} a_{ij}^2$, where n is the number of points and a_{ij} is the distance between the i-th and j-th points.

b) Prove that the moment of inertia with respect to the center of mass of the system of points whose masses are equal to $m_1, \dots, m_n$ is equal to $\frac{1}{m}\sum_{i<j} m_im_ja_{ij}^2$, where $m = m_1 + \dots + m_n$ and a_{ij} is the distance between the i-th and j-th points.

14.11. Prove that the sum of squared lengths of a tetrahedron's medians is equal to $\frac{4}{9}$ of the sum of squared lengths of its edges.

14.12. Unit masses are placed at the vertices of a tetrahedron. Prove that the moment of inertia of this system relative to the center of mass is equal to the sum of squared distances between the midpoints of the opposite edges of tetrahedron.

14.13. Triangle ABC is given. Find the locus of points X in space such that $XA^2 + XB^2 = XC^2$.

14.14. Two triangles, an equilateral one with side a and an isosceles right one with legs equal to b are placed in space so that their centers of mass coincide. Find the sum of squared distances from all the vertices of one of the triangles to all the vertices of another triangle.

14.15. Inside a sphere of radius R, n points are fixed. Prove that the sum of the squared pairwise distances between these points does not exceed n^2R^2.

14.16. Points $A_1, \dots, A_n$ lie on one sphere and M is their center of mass. Lines $MA_1, \dots, MA_n$ intersect this sphere at points $B_1, \dots, B_n$ (distinct from $A_1, \dots, A_n$). Prove that

$$MA_1 + \dots + MA_n \leq MB_1 + \dots + MB_n.$$

§3. Barycentric coordinates

Tetrahedron $A_1A_2A_3A_4$ is given in space. If point X is the center of mass of the vertices of this tetrahedron whose masses are m_1, m_2, m_3 and m_4, respectively, then the quadruple (m_1, m_2, m_3, m_4) is called the *barycentric coordinates* of point X relative the tetrahedron $A_1A_2A_3A_4$.

14.17. Tetrahedron $A_1A_2A_3A_4$ in space is given.

a) Prove that any point X has certain barycentric coordinates relative the given tetrahedron.

b) Prove that the barycentric coordinates of point X are uniquely defined if

$$m_1 + m_2 + m_3 + m_4 = 1.$$

14.18. In barycentric coordinates relative to tetrahedron $A_1A_2A_3A_4$ find the equation of: a) line A_1A_2; b) plane $A_1A_2A_3$; c) the plane that passes through A_3A_4 parallel to A_1A_2.

14.19. Prove that if points whose barycentric coordinates are (x_i) and (y_i) belong to some plane then the point with barycentric coordinates $(x_i + y_i)$ also belongs to the same plane.

14.20. Let S_a, S_b, S_c and S_d be the areas of faces BCD, ACD, ABD and ABC, respectively, of tetrahedron $ABCD$. Prove that in the system of barycentric coordinates relative this tetrahedron $ABCD$:

a) the coordinates of the center of the inscribed sphere are (S_a, S_b, S_c, S_d);

b) the coordinates of the center of the escribed sphere tangent to face ABC are $(S_a, S_b, S_c, -S_d)$.

14.21. Find the equation of the sphere inscribed in tetrahedron $A_1A_2A_3A_4$ in barycentric coordinates related to it.

14.22. a) Prove that if the centers I_1, I_2, I_3 and I_4 of escribed spheres tangent to the faces of a tetrahedron lie on its circumscribed sphere, then this tetrahedron is an equifaced one.

b) Prove that the converse is also true: for an equifaced tetrahedron points I_1, I_2, I_3 and I_4 lie on the circumscribed sphere.

Solutions

14.1. Let X and O be arbitrary points in plane. Then

$$m_1\{OX_1\}+\dots+m_n\{OX_n\} = (m_1+\dots+m_n)\{OX\}+m_1\{XX_1\}+\dots+m_n\{XX_n\}$$

and, therefore, point O is the center of mass of the given system of points if and only if

$$(m_1 + \dots + m_n)\{OX\} + m_1\{XX_1\} + \dots m_n\{XX_n\} = \{0\},$$

i.e.,

$$\{XO\} = \frac{1}{m_1 + \dots + m_n} \cdot (m_1\{XX_1\} + \dots + m_n\{XX_n\}).$$

This argument implies the solution of both headings of the problem.

14.2. Let Z be an arbitrary point, $a = a_1 + \dots + a_n$ and $b = b_1 + \dots + b_m$. Then $\{ZX\} = \frac{1}{a}(a_1\{ZX_1\} + \dots + a_n\{ZX_n\})$ and $\{ZY\} = \frac{1}{b}(b_1\{ZY_1\} + \dots + b_m\{ZY_m\})$. If O is the center of mass of the two points - X with mass a and Y with mass b - then

$$\{ZO\} = \frac{1}{a+b}(a\{ZX\} + b\{ZY\}) = \\ \frac{1}{a+b}(a_1\{ZX_1\} + \dots + a_n\{ZX_n\} + b_1\{ZY_1\} + \dots + b_m\{ZY_m\}),$$

i.e., O is the center of mass of the system of points $X_1, \dots, X_n, Y_1, \dots, Y_m$ with masses $a_1, \dots, a_n, b_1, \dots, b_m$, respectively.

14.3. Let us place unit masses in the vertices of the tetrahedron. The center of mass of these points lies on the segment that connects the vertex of the tetrahedron with the center of mass of the vertices of the opposite face and divides this segment in the ratio 3 : 1 counting from the vertex. Therefore, all the medians of the tetrahedrons pass through its center of mass.

The center of mass of the tetrahedron also lies on the segment that connects the centers of mass of opposite edges (i.e., their midpoints) and divides this segment in halves.

14.4. Let us place unit masses at points A_1, B and D. Let O be the center of mass of this system. Then

$$3\{AO\} = \{AA_1\} + \{AB\} + \{AD\} = \{AA_1\} + \{A_1B_1\} + \{B_1C_1\} = \{AC_1\},$$

i.e., point O lies on diagonal AC_1. On the other hand, the center of mass of points A_1, B and D lies in plane A_1BD, hence, $O = M$ and, therefore, $3\{AM\} = 3\{AO\} = \{AC_1\}$.

14.5. Let us place unit masses at points A, B, C, A_1, B_1 and C_1. On the one hand, the center of mass of this system coincides with the center of mass of the triangle with vertices at the midpoints of segments AA_1, BB_1 and CC_1.

On the other hand, it coincides with the midpoint of the segment that connects the center of mass X of points A_1, B_1 and C_1 with the center of mass M of triangle ABC. Point M is fixed and point X moves along line l. Therefore, the midpoint of segment MX lies on the line homothetic to line l with center M and coefficient 0.5.

14.6. Let us place points of mass 1, p, pq and q at points A, B, C and D, respectively, and consider the center of mass P of this system of points. Since K is the center of mass of points A and B, M is the center of mass of points C and D, it follows that point P lies on segment KM, where

$$KP : PM = (pq + q) : (1 + p) = q.$$

Similarly, point P lies on segment LN and $NP : PL = p$.

14.7. Let M be the center of mass of tetrahedron $ABCD$. Then

$$\begin{gathered}\{MA_1\} + \{MB_1\} + \{MC_1\} + \{MD_1\} = \\ (\{MA\} + \{MB\} + \{MC\} + \{MD\}) + (\{AA_1\}+ \\ \{BB_1\} + \{CC_1\} + \{DD_1\}) = \\ \{AA_1\} + \{BB_1\} + \{CC_1\} + \{DD_1\}.\end{gathered}$$

Vectors $\{AA_1\}$, $\{BB_1\}$, $\{CC_1\}$ and $\{DD_1\}$ are perpendicular to the tetrahedron's faces and their lengths are proportional to the areas of the faces (this follows from the fact that the areas of the tetrahedron's faces are inverse proportional to the lengths of the heights dropped onto them). Therefore, the sum of these vectors is equal to zero (cf. Problem 7.19), hence, M is the center of mass of tetrahedron $A_1B_1C_1D_1$.

14.8. a) Since

$$\{MA_1\} + \dots + \{MA_n\} = \{MB_1\} + \dots + \{MB_n\} = \{0\},$$

we see that by adding equalities $\{MA_i\} + \{A_iB_i\} + \{B_iN\} = \{MN\}$ for all $i = 1, \dots, n$ we get $\{A_1B_1\} + \dots + \{A_nB_n\} = n\{MN\}$. Therefore, segment MN is parallel to the edges of the prism and $\{A_1B_1\} + \dots + \{A_nB_n\} = nMN$.

Notice also that if instead of polygon $B_1 \dots B_n$ we take one of the bases of the prism, we see that line MN passes through the centers of the prism's bases.

b) Let us divide the base of the prism into triangles by connecting its center with the vertices; the areas of these triangles are equal. Considering the triangular prisms whose bases are the obtained triangles we can divide the given part of the prism into the polyhedrons with triangular bases and parallel lateral edges. By Problem 3.24 the volumes of these polyhedrons are equal to $\frac{s(A_1B_1+A_2B_2+MN)}{3n}$, $\dots$, $\frac{s(A_nB_n+A_1B_1+MN)}{3n}$. Therefore, the volume of the whole part of the prism confined between the given planes is equal to

$$\frac{s(2(A_1B_1+\dots+A_nB_n)+nMN)}{3n}.$$

It remains to notice that

$$A_1B_1+\dots+A_nB_n = nMN.$$

14.9. Let us enumerate the points of the given system. Let $\mathbf{x}_i$ be the vector with the beginning at O and the endpoint at the i-th point; let the mass of this point be equal to m_i. Then $\sum m_i\mathbf{x}_i = \mathbf{0}$. Further, let $\mathbf{a} = \{XO\}$. Then $I_O = \sum m_i x_i^2$, and

$$\begin{aligned} I_X &= \textstyle\sum m_i(\mathbf{x}_i+a)^2 = \sum m_i x_i^2 + 2(\sum m_i\mathbf{x}_i, \mathbf{a}) + \sum m_i a^2 = \\ &= I_O + ma^2. \end{aligned}$$

14.10. a) Let $\mathbf{x}_i$ be the vector with the beginning at the center of mass, O, and the endpoint at the i-th point. Then

$$\sum_{i,j}(\mathbf{x}_i-\mathbf{x}_j)^2 = \sum_{i,j}(x_i^2+x_j^2) - 2\sum_{i,j}(\mathbf{x}_i,\mathbf{x}_j),$$

where sum runs over all possible pairs of the point's numbers. Clearly,

$$\sum_{i,j}(x_i^2+x_j^2) = 2n\sum_i x_i^2 = 2nI_O \sum_{i,j}(\mathbf{x}_i,\mathbf{x}_j) = \sum_i(\mathbf{x}_i,\sum_j\mathbf{x}_j) = 0.$$

Therefore,

$$2nI_O = \sum_{i,j}(\mathbf{x}_i-\mathbf{x}_j)^2 = 2\sum_{i<j}a_{ij}^2.$$

b) Let $\mathbf{x}_i$ be the vector with the beginning at the center of mass, O and the endpoint at the i-th point. Then

$$\sum_{i,j}m_im_j(\mathbf{x}_i-\mathbf{x}_j)^2 = \sum_{i,j}m_im_j(x_i^2+x_j^2) - 2\sum_{i,j}m_im_j(\mathbf{x}_i,\mathbf{x}_j).$$

Clearly,

$$\sum_{i,j}m_im_j(x_i^2+x_j^2) =$$

$$\sum_i m_i\sum_j(m_jx_i^2+m_jx_j^2) =$$

$$\sum_i m_i(mx_i^2+I_O) = 2mI_O$$

and

$$\sum_{i,j} m_i m_j(\mathbf{x}_i, \mathbf{x}_j) = \sum m_i(\mathbf{x}_i, \sum_j m_j \mathbf{x}_j) = 0.$$

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Therefore,

$$2mI_O = \sum_{i,j} m_i m_j (\mathbf{x}_i - \mathbf{x}_j)^2 = 2\sum_{i<j} m_i m_j a_{ij}^2.$$

14.11. Let us place unit masses at the vertices of the tetrahedron. Since their center of mass — the intersection point of the tetrahedron's medians — divides each median in ratio 3 : 1, the moment of inertia of the tetrahedron relative the center of mass is equal to

$$(\frac{3}{4}m_a)^2 + \dots + (\frac{3}{4}m_d)^2 = \frac{9}{16}(m_a^2 + m_b^2 + m_c^2 + m_d^2).$$

On the other hand, by Problem 14.10 it is equal to the sum of squares of the length of the tetrahedron's edges divided by 4.

14.12. The center of mass O of tetrahedron $ABCD$ is the intersection point of segments that connect the midpoints of the opposite edges of the tetrahedron and point O divides each of these segments in halves (Problem 14.3 b)). If K is the midpoint of edge AB, then

$$AO^2 + BO^2 = 2OK^2 + \frac{AB^2}{2}.$$

Let us write such equalities for all edges of the tetrahedron and take their sum. Since from each vertex 3 edges exit, we get $3I_O$ in the left-hand side. If L is the midpoint of segment CD, then $2OK^2 + 2OL^2 = KL^2$. Moreover, as follows from Problem 14.10 a), the sum of the squared lengths of the terahedron's edges is equal to $4I_O$. Therefore, in the right-hand side of the equality we get $d + 2I_O$, where d is the sum of the squared distances between the midpoints of the opposite edges of the tetrahedron. After simplification we get the desired statement.

14.13. Place unit masses at vertices A and B and mass -1 at vertex C. The center of mass, M, of this system of points is a vertex of parallelogram $ACBM$. By the hypothesis

$$I_X = XA^2 + XB^2 - XC^2 = 0$$

and, since

$$I_X = (1 + 1 - 1)MX^2 + I_M$$

(Problem 14.9), it follows that

$$MX^2 = -I_M = a^2 + b^2 - c^2,$$

where a, b and c are the lengths of the sides of triangle ABC (Problem 14.10 b)). Thus, if $\angle C < 90°$, then the locus we seek for is the sphere of radius $\sqrt{a^2 + b^2 - c^2}$ centered at M.

14.14. If M is the center of mass of triangle ABC, then

$$I_M = \frac{AB^2 + BC^2 + AC^2}{3}$$

(cf. Problem 14.10 a)) and, therefore, for any point X we have

$$XA^2 + XB^2 + XC^2 = I_X = 3XM^2 + I_M = 3XM^2 + \frac{AB^2 + BC^2 + AC^2}{3}.$$

If ABC is the given right triangle, $A_1B_1C_1$ is the given equilateral triangle and M is their common center of mass, then

$$A_1A^2 + A_1B^2 + A_1C^2 = 3A_1M^2 + \frac{4b^2}{3} = a^2 + \frac{4b^2}{3}.$$

Write similar equalities for points B_1 and C_1 and take their sum. We deduce that the desired sum of the squares is equal to $3a^2 + 4b^2$.

14.15. Let us place unit masses in the given points. As follows from the result of Problem 14.10 a)), the sum of squared pairwise distances between these points is equal to nI, where I is the moment of inertia of the system of points relative its center of mass. Now, let us consider the moment of inertia of the system relative the center O of the sphere.

On the one hand, $I \leq I_O$ (cf. Problem 14.9). On the other hand, since the distance from point O to any of the given points does not exceed R, we have $I_O \leq nR^2$. Therefore, $nI \leq n^2R^2$ and the equality is only attained if $I = I_O$ (i.e., the center of mass coincides with the center of the sphere) and $I_O = nR^2$ (i.e., all the points lie on the surface of the given sphere).

14.16. Let O be the center of the given sphere. If chord AB passes through point M, then $AM \cdot BM = R^2 - d^2$, where $d = MO$. Denote by I_X the moment of inertia of the system of points $A_1, \dots, A_n$ relative point X. Then $I_O = I_M + nd^2$ by Problem 14.9. On the other hand, since $OA_i = R$, then $I_O = nR^2$. Therefore,

$$A_iM \cdot B_iM = R^2 - d^2 = \frac{1}{n}(A_1M^2 + \dots + A_nM^2).$$

Thus, if we set $a_i = A_iM$, then the required inequality takes the form

$$a_1 + \dots + a_n \leq \frac{1}{n}(a_1^2 + \dots + a_n^2)(\frac{1}{a_1} + \dots + \frac{1}{a_n}).$$

To prove this inequality we should make use of the inequality

$$x + y \leq \frac{x^2}{y} + \frac{y^2}{x}$$

which is obtained from the inequality $xy \leq x^2 - xy + y^2$ by multiplying both of its sides by $\frac{x+y}{xy}$.

14.17. Denote: $\mathbf{e}_1 = \{A_4A_1\}$, $\mathbf{e}_2 = \{A_4A_2\}$, $\mathbf{e}_3 = \{A_4A_3\}$ and $\mathbf{x} = \{XA_4\}$. Point X is the center of mass of the vertices of tetrahedron $A_1A_2A_3A_4$ with masses m_1, m_2, m_3 and m_4, respectively, if and only if

$$m_1(\mathbf{x} + \mathbf{e}_1) + m_2(\mathbf{x} + \mathbf{e}_2) + m_3(\mathbf{x} + \mathbf{e}_3) + m_4\mathbf{x} = \mathbf{0},$$

i.e.,

$$m\mathbf{x} = -(m_1\mathbf{e}_1 + m_2\mathbf{e}_2 + m_3\mathbf{e}_3), \text{ where } m = m_1 + m_2 + m_3 + m_4.$$

Let us assume that $m = 1$. Any vector $\mathbf{x}$ can be represented in the form $\mathbf{x} = -m_1\mathbf{e}_1 - m_2\mathbf{e}_2 - m_3\mathbf{e}_3$, where numbers m_1, m_2 and m_3 are uniquely defined. The number m_4 is found from the formula $m_4 = 1 - m_1 - m_2 - m_3$.

14.18. The point whose barycentric coordinates are (x_1, x_2, x_3, x_4) lies:

a) on line A_1A_2 if $x_3 = x_4 = 0$;

b) in plane $A_1A_2A_3$ if $x_4 = 0$.

c) Let us make use of notations of Problem 14.17. Point X lies in the indicated plane if $\mathbf{x} = \lambda(\mathbf{e}_1 - \mathbf{e}_2) + \mu\mathbf{e}_3$, i.e., $x_1 = -x_2$.

14.19. The point whose barycentric coordinates are $(x_i + y_i)$ is the center of mass of points whose barycentric coordinates are (x_i) and (y_i). It is also clear that the center of mass of two points lies on the line that passes through them.

14.20. a) The center of the inscribed sphere is the intersection point of the bisector planes of the dihedral angles of the tetrahedron. Let M be the intersection point of edge AB with the bisector plane of the dihedral angle at edge CD. Then $AM : NB = S_b : S_a$ (Problem 3.32) and, therefore, the barycentric coordinates of point M are equal to $(S_a, S_b, 0, 0)$. The bisector plane of the dihedral angle at edge CD passes through the point with coordinates $(S_a, S_b, 0, 0)$ and through line CD the coordinates of whose points are $(0, 0, x, y)$. Therefore, this plane consists of points whose coordinates are (S_a, S_b, x, y), cf. Problem 14.19. Thus, point (S_a, S_b, S_c, S_d) belongs to the bisector plane of the dihedral angle at edge CD. We similarly prove that it belongs to the other bisector plane.

b) The center of the escribed sphere tangent to face ABC is the intersection point of the bisector planes of the dihedral angles at edges AD, BD, CD and the bisector planes of the exterior dihedral angles at edges AB, BC, CA. Let M be the intersection point of the extension of edge CD with the bisector plane of the exterior angle at edge AB (if this bisector plane is parallel to edge CD, then we have to make use of the result of Problem 14.18 c)). The same arguments as in the solution of Problem 3.32 show that $CM : MD = S_d : S_c$. The subsequent part of the proof is the same as that of the preceding problem.

14.21. Let X be an arbitrary point, O the center of the sphere circumscribed about the given tetrahedron, $\mathbf{e}_i = \{OA_i\}$ and $\mathbf{a} = \{XO\}$. If the barycentric coordinates of point X are (x_1, x_2, x_3, x_4), then

$$\sum x_i(\mathbf{a} + \mathbf{e}_i) = \sum x_i\{XA_i\} = \mathbf{0},$$

because X is the center of mass of points $A_1, \dots, A_4$ whose masses are $x_1, \dots, x_4$, respectively. Hence, $(\sum x_i)\mathbf{a} = -\sum x_i\mathbf{e}_i$. Point X lies on the sphere circumscribed about the tetrahedron if and only if $|\mathbf{a}| = XO = R$, where R is the radius of the sphere. Therefore, the circumscribed sphere of the tetrahedron is given in the barycentric coordinates by the equation

$$R^2(\sum x_i)^2 = (\sum x_i\mathbf{e}_i)^2,$$

i.e.,

$$R^2\sum x_i^2 + 2R^2\sum_{i<j} x_ix_j = R^2\sum x_i^2 + 2\sum_{i<j} x_ix_j(\mathbf{e}_i, \mathbf{e}_j)$$

because $|\mathbf{e}_i| = R$. This equation can be rewritten in the form

$$\sum_{i<j} x_ix_j(R^2 - (\mathbf{e}_i,\ \mathbf{e}_j)) = 0.$$

Now, notice that $2(R^2 - (\mathbf{e}_i, \mathbf{e}_j)) = a_{ij}^2$, where a_{ij} is the length of edge A_iA_j. Indeed,

$$a_{ij}^2 = |\mathbf{e}_i - \mathbf{e}_j|^2 = |\mathbf{e}_i|^2 + |\mathbf{e}_j|^2 - 2(\mathbf{e}_i, \mathbf{e}_j) = 2(R^2 - (\mathbf{e}_i, \mathbf{e}_j)).$$

As a result we see that the sphere circumscribed about tetrahedron $A_1A_2A_3A_4$ is given in barycentric coordinates by equation $\sum_{i<j} x_ix_ja_{ij} = 0$, where a_{ij} is the length of edge A_iA_j.

14.22. a) Let S_1, S_2, S_3 and S_4 be areas of faces $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$ and $A_1A_2A_3$, respectively. The barycentric coordinates of points I_1, I_2, I_3 and I_4 are $(-S_1, S_2, S_3, S_4)$, $(S_1, -S_2, S_3, S_4)$, $(S_1, S_2, -S_3, S_4)$ and $(S_1, S_2, S_3, -S_4)$ (Problem 14.20 b)) and the equation of the circumscribed sphere of the tetrahedron in barycentric coordinates is $\sum_{i<j} a_{ij}^2 x_ix_j = 0$, where a_{ij} is the length of edge A_iA_j (Problem 14.21).

Let us express the conditions of membership of points I_1 and I_2 to the circumscribed sphere (for simplicity we have denoted $a_{ij}^2 S_iS_j$ by y_{ij}):

$$y_{12} + y_{13} + y_{14} = y_{23} + y_{34} + y_{24};$$

$$y_{12} + y_{23} + y_{24} = y_{13} + y_{34} + y_{14}.$$

Adding up these equalities we get $y_{12} = y_{34}$. Similarly, adding up such equalities for points I_i and I_j we get $y_{ij} = y_{kl}$, where the set of numbers $\{i, j, k, l\}$ coincides with a permutation of the set $\{1, 2, 3, 4\}$.

By multiplying the equalities $y_{13} = y_{23}$ and $y_{14} = y_{24}$ we get $y_{13}y_{14} = y_{23}y_{24}$, i.e.,

$$S_1S_3a_{13}^2S_1S_4a_{14}^2 = S_2S_3a_{23}^2S_2S_4a_{24}^2.$$

Since all the numbers S_i and a_{ij} are positive, it follows that $S_1a_{13}a_{14} = S_2a_{23}a_{24}$, i.e., $\frac{a_{23}a_{24}}{S_1} = \frac{a_{13}a_{14}}{S_2}$. By multiplying both sides of the equality by a_{34} we get

$$\frac{a_{23}a_{24}a_{34}}{S_1} = \frac{a_{13}a_{14}a_{34}}{S_2}.$$

Each side of this equality is the ratio of the product of the length of the triangle's sides to its area. It is easy to verify that such a ratio is equal to 4 times the radius of the circle circumscribed about the triangle. Indeed, $S = \frac{1}{2}ab\sin\gamma = \frac{abc}{4R}$. Therefore, the radii of the circles circumscribed about faces $A_2A_3A_4$ and $A_1A_3A_4$ are equal.

We similarly prove that the radii of all the faces of the tetrahedron are equal. Now, it remains to make use of the result of Problem 6.25 c).

b) Let us make use of the notations of the preceding problem. For an equifaced tetrahedron $S_1 = S_2 = S_3 = S_4$. Therefore, the fact that point I_1 belongs to the circumscribed sphere of the tetrahedron take the form

$$a_{12} + a_{13} + a_{14} = a_{23} + a_{34} + a_{24}.$$

This equality follows from the fact that $a_{12} = a_{34}$, $a_{13} = a_{24}$ and $a_{14} = a_{23}$. We similarly verify that points I_2, I_3 and I_4 belong to the circumscribed sphere.

Remark. In the solution of Problem 6.32 the statement of heading b) is proved by another method.

CHAPTER 15. MISCELLANEOUS PROBLEMS

§1. Examples and counterexamples

15.1. a) Does there exist a quadrilateral pyramid two nonadjacent faces of which are perpendicular to the plane of the base?

b) Does there exist a hexagonal pyramid whose three (it is immaterial whether they are adjacent or not) lateral faces are perpendicular to the plane of the base?

15.2. Vertex E of tetrahedron $ABCD$ lies inside tetrahedron $ABCD$. Is it necessary that the sum of the lengths of edges of the outer tetrahedron is greater than the sum of the lengths of edges of the inner tetrahedron?

15.3. Does there exist a tetrahedron all faces of which are acute triangles?

15.4. Does there exist a tetrahedron the basis of all whose heights lie outside the corresponding faces?

15.5. In pyramid $SABC$ edge SC is perpendicular to the base. Can angles ASB and ACB be equal?

15.6. Is it possible to intersect an arbitrary trihedral angle with a plane so that the section is an equilateral triangle?

15.7. Find the plane angles at the vertices of a trihedral angle if it is known that any section of the latter is an acute triangle.

15.8. Is it possible to place 6 pairwise nonparallel lines in space so that all the pairwise angles between them are equal?

15.9. Is it necessary that a polyhedron all whose faces are equal squares must be a cube?

15.10. All the edges of a polyhedron are equal and tangent to one sphere. Is it necessary that its vertices lie on one sphere?

15.11. Can a finite set of points in space not in one plane possess the following property: for any two points A and B from this set there are two more points C and D from this set such that $AB \parallel CD$ and these lines do not coincide?

15.12. Is it possible to place 8 nonintersecting tetrahedrons so that any two of them touch each other along a piece of surface with nonzero area?

§2. Integer lattices

The set of points in space all the three coordinates of which are integers is called an *integer lattice* and the points themselves the *nodes* of the integer lattice. The planes parallel to the coordinate planes and passing through the nodes of an integer lattice divide the space into unit cubes.

15.13. Nine vertices of a convex polyhedron lie at nodes of an integer lattice. Prove that either inside it or on its lattice there is one more node of an integer lattice.

15.14. a) For what n there exists a regular n-gon with vertices in nodes of a (spatial) integer lattice?

b) What regular polyhedrons can be placed so that their vertices lie in nodes of an integer lattice?

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15.15. Is it possible to draw a finite number of planes in space so that at least one of these planes would intersect each small cube of the integer lattice?

15.16. Prove that among parallelograms whose vertices are at integer points of the plane $ax + by + cz = 0$, where a, b and c are integers, the least area S is equal to the least length l of the vector with integer coordinates perpendicular to this plane.

15.17. Vertices A_1, B, C_1 and D of cube $ABCDA_1B_1C_1D_1$ lie in nodes of an integer lattice. Prove that its other vertices also lie in nodes of an integer lattice.

15.18. a) Given a parallelepiped (not necessarily a rectangular one) with vertices in nodes of an integer lattice such that a nodes of the lattice are inside it, b nodes are inside its faces and c nodes are inside its edges. Prove that its volume is equal to $1 + a + \frac{1}{2}b + \frac{1}{4}c$.

b) Prove that the volume of the tetrahedron whose only integer points are its vertices can be however great.

§3. Cuttings. Partitions. Colourings

15.19. a) Cut a tetrahedron with edge $2a$ into tetrahedrons and octahedrons with edge a.

b) Cut an octahedron with edge $2a$ into tetrahedrons and octahedrons with edge a.

15.20. Prove that the space can be filled in with regular tetrahedrons and octahedrons without gaps.

15.21. Cut a cube into three equal pyramids.

15.22. Into what minimal number of tetrahedrons can a cube be cut?

15.23. Prove that any tetrahedron can be cut by a plane into two parts so that one can compose the same tetrahedron from them by connecting them not as they were connected before but in a new way.

15.24. Prove that any polyhedron can be cut into convex polyhedrons.

15.25. a) Prove that any convex polyhedron can be cut into tetrahedrons.

b) Prove that any convex polyhedron can be cut into tetrahedrons whose vertices lie in vertices of a polyhedron.

* * *

15.26. Into how many parts is the space divided by the planes of faces of: a) a cube; b) a tetrahedron?

15.27. Into what greatest number of parts can the sphere be divided by n circles?

15.28. Given n planes in space so that any three of them have exactly one common point and no four of them pass through one point, prove that they divide the space into $\frac{1}{6}(n^2 + 5n + 6)$ parts.

15.29. Given n $(n \geq 5)$ planes in space so that any three of them have exactly one common point and no four of them pass through one point, prove that among the parts into which these planes divide the space there are not less than $\frac{1}{4}(2n - 3)$ tetrahedrons.

* * *

15.30. A stone is of the shape of a regular tetrahedron. This stone is rolled over the plane by rotating about its edges. After several such rotations the stone returns to the initial position. Can its faces change places?

15.31. A rectangular parallelepiped of size $2l \times 2m \times 2n$ is cut into unit cubes and each of these cubes is painted one of 8 colours so that any two cubes with at least one common vertex are painted different colours. Prove that all the corner cubes are differently painted.

§4. Miscellaneous problems

15.32. A plane intersects the lower base of a cylinder along a diameter and has only one common point with the cylinder's upper base. Prove that the area of the cut off part of the lateral surface of the cylinder is equal to the area of its axial section.

15.33. Given $3(2^n - 1)$ points inside a convex polyhedron of volume V. Prove that the polyhedron contains another polyhedron of volume $\frac{V}{2^n}$ whose internal part contains none of the given points.

15.34. Given 4 points in space not in one plane. How many distinct parallelepipeds for which these points are vertices are there?

Solutions

15.1. Yes, such pyramids exist. For their bases we can take, for instance, a quadrilateral and a nonconvex hexagon plotted on Fig. 106 the vertices of these pyramids on the perpendiculars raised at points P and Q, respectively.

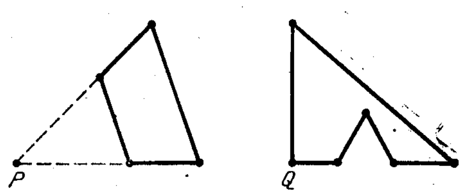

FIGURE 106 (SOL. 15.1)

15.2. No, not necessarily. Let us consider an isosceles triangle ABC whose base AC is much shorter than its lateral side. Let us place vertex D close to the midpoint of side AC and vertex E inside tetrahedron $ABCD$ close to vertex B. The perimeter of the outer tetrahedron can be made however close to $3a$, where a is the length of the lateral side of triangle ABC and the perimeter of the inner one however close to $4a$.

15.3. Yes, there is. Let angle C of triangle ABC be obtuse, point D lie on the height dropped from vertex C. By slightly raising point D over the plane ABC we get the desired tetrahedron.

15.4. Yes, it exists. A tetrahedron two opposite dihedral angles of which are obtuse possesses this property. To construct such a tetrahedron we can, for example, take two diagonals of a square and slightly lift one of them over the other.

REMARK. The base of the shortest height of any tetrahedron lies inside the triangle whose sides pass through vertices of the opposite face parallelly with its edges (cf. Problem 12.16).

15.5. Yes, they can. Let points C and S lie on one arc of a circle that passes through A and B so that $SC \perp AB$ and point C is closer to line AB than point S is (see Fig. 107). Then we can rotate triangle ABS about AB so that segment SC becomes perpendicular to plane ABC.

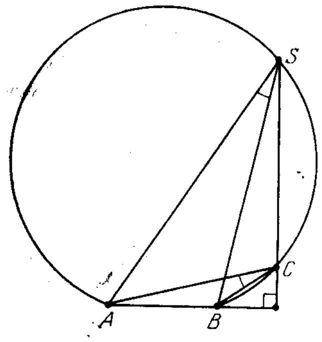

FIGURE 107 (SOL. 15.5)

15.6. No, not for every angle. Let us consider a trihedral angle $SABC$ for which $\angle BSC < 60°$ and edge AS is perpendicular to face SBC. Suppose that its section ABC is an equilateral triangle. In right triangles ABS and ACS the hypothenuses are equal because $SB = SC$. In isosceles triangle SBC, the angle at vertex S is the smallest, hence, $BC < SB$. It is also clear that $SB < AB$ and, therefore, $BC < AB$. Contradiction.

15.7. First, let us prove that any section of the trihedral angle with right planar angles is an acute triangle. Indeed, let the intersecting plane cut off the edges segments of length a, b and c. Then the squares of the lengths of the sides of the section are equal to $a^2 + b^2$, $b^2 + c^2$ and $a^2 + c^2$. The sum of squares of any two sides is greater than the square of the third one and, therefore, the triangle is an acute one.

Now, let us prove that if all the planar angles of the trihedral angle are right ones then it has a section: an acute triangle. If the trihedral angle has an acute plane angle, then on the leg of this trihedral angle draw equal segments SA and SB; if point C on the third edge is taken sufficiently close to vertex S, then triangle ABC is an acute one.

If the trihedral angle has an acute plane angle, then we can select points A and B on the legs of this trihedral angle , so that the angle $\angle SAB$ is an obtuse one; and if point C on the third leg is taken sufficiently close to vertex S, then triangle ABC is an acute one.

15.8. Yes, it is possible. Let us draw lines that connect the center of the icosahedron with its vertices (cf. Problem 9.4). It is easy to verify that any two such lines pass through two points that are the endpoints of one edge.

15.9. No, not necessarily. Let us take a cube and glue equal cubes to each of its faces. All the faces of the obtained (nonconvex) polyhedron are equal squares.

15.10. No, not necessarily. On the faces of a cube as on bases, construct regular quadrangular pyramids with dihedral angles at the bases equal to 45°. As a result we get a 12-hedron with 14 vertices of which 8 are vertices of the cube and 6 are vertices of the constructed pyramids; the edges of the cube are diagonals of its faces and, therefore, cannot serve as its edges.

All the edges of the constructed polyhedron are equal and equidistant from the center of the cube. All the vertices of the polyhedron cannot belong to one sphere since the distance from the vertices of the cube to the center is equal to $\frac{\sqrt{3}}{2}a$, where a is the edge of the cube whereas the distance of the other vertices from the center of the cube is equal to a.

15.11. Yes, it can. It is easy to verify that the vertices of a regular hexagon possess the desired property. Now, consider two regular hexagons with a common center O but lying in distinct planes. If A and B are vertices of distinct hexagons, then we can take for C and D points symmetric to A and B, respectively, through O.

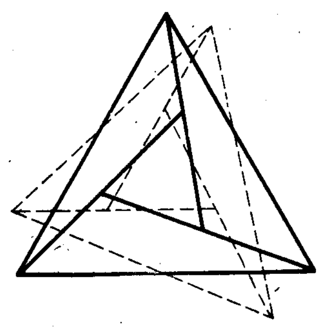

FIGURE 108 (SOL. 15.12)

15.12. Yes, this is possible. On Fig. 108 the solid line plots 4 triangles of which one lies inside other three. Let us consider four triangular pyramids with a common vertex whose bases are these triangles. We similarly construct four more triangular pyramids with a common vertex (that lie on the other side of the plot's plane) whose bases are the triangles plotted by dashed lines. The obtained 8 tetrahedrons possess the required property.

15.13. Each of the three coordinates of a node of an integer lattice can be either even or odd; altogether $2^3 = 8$ distinct possibilities. Therefore, among nine vertices of a polyhedron there are two vertices with coordinates of the same parity. The midpoint of the segment that connects these vertices has integer coordinates.

15.14. a) First, let us prove that for $n = 3, 4, 6$ there exists a regular n-gon with vertices in nodes of an integer lattice. Let us consider cube $ABCDA_1B_1C_1D_1$ the coordinates of whose vertices are equal to $(\pm1, \pm1, \pm1)$. Then the midpoints of edges AB, BC, CC_1, C_1D_1, D_1A_1 and A_1A are the vertices of a regular hexagon and all of them have integer coordinates (Fig. 109); the midpoints of edges AB, CC_1 and D_1A_1 are the vertices of an equilateral triangle; it is also clear that $ABCD$ is a square whose vertices have integer coordinates.

Now, let us prove that for $n \neq 3, 4, 6$ there is no regular n-gon with vertices in nodes of an integer lattice. Suppose, contrarywise that for some $n \neq 3, 4, 6$ such an n-gon exists. Among all the n-gons with vertices in nodes of the lattice we can select one with the shortest side.

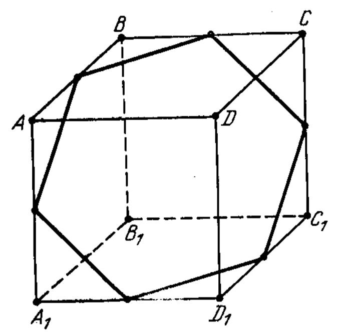

FIGURE 109 (SOL. 15.14)

To prove it, let us verify that the length of a side of such an n-gon can only take finitely many values smaller than the given one. It remains to notice that the length of any segment with the endpoints in nodes of the lattice is equal to $\sqrt{n_1^2+n_2^2+n_3^2}$, where n_1, n_2 and n_3 are integers.

Let $A_1A_2\dots A_n$ be the chosen n-gon with the shortest side. Let us consider a regular n-gon $B_1\dots B_n$, where point B_i is obtained from point A_i by translation by vector $\{A_{i+1}A_{i+2}\}$, i.e., $\{A_iB_i\}=\{A_{i+1}A_{i+2}\}$. Since the translation by vector with integer coordinates sends a node of the lattice to a node of the lattice, B_i is a node of the lattice.

In order to get a contradiction it remains to prove that the length of a side of polygon $B_1\dots B_n$ is strictly shorter than a side of polygon $A_1\dots A_n$ (and is not equal to zero). The proof of this is quite obvious; we only have to consider separately two cases: $n=5$ and $n\geq 7$.

b) First, let us prove that a cube, a regular tetrahedron and an octahedron can be placed in the desired way. To this end consider cube $ABCDA_1B_1C_1D_1$ the coordinates of whose vertices are $(\pm1,\pm1,\pm1)$. Then AB_1CD_1 is the required tetrahedron and the midpoints of the faces of the considered cube are vertices of the required octahedron.

Now, let us prove that neither dodecahedron nor icosahedron can be placed in the desired way. As follows from the preceding problem, there is no regular pentagon with vertices in nodes of the lattice. It remains to verify that both dodecahedron and icosahedron have a set of vertices that single out a regular pentagon.

For a dodecahedron these are vertices of any of the faces and for the icosahedron these are vertices which are endpoints of the edges that go out of one of the vertex.

15.15. No, this is impossible. Let n planes be given in space. If a small cube of the lattice intersects with a plane, then it lies entirely inside a band of width $2\sqrt{3}$ consisting of all the points whose distance from the given plane is not greater than $\sqrt{3}$ ($\sqrt{3}$ is the greatest distance between points of a small cube).

Let us consider a ball of radius R. If all the small cubes of the lattice having a common point with this ball intersect with given planes then the slices of width $2\sqrt{3}$ determined by given planes fill in the whole ball. The volume of the part of each of such slice that lies inside the ball does not exceed $2\sqrt{3}\pi R^2$. Since the volume of the ball does not exceed the sum of the volumes of the slices,

$$\frac{4\pi R^3}{3}\leq 2\sqrt{3}n\pi R^2,\quad \text{i.e.,}\quad R\leq\frac{3\sqrt{3}}{2}n.$$

Therefore, if $R > \frac{3\sqrt{3}}{2}n$, then n planes cannot intersect all the small cubes of the lattice that have common points with a ball of radius R.

15.16. We can assume that numbers a, b and c are relatively prime, i.e., the largest number that divides all of them is equal to 1. The coordinates of a vector perpendicular to this plane are $(\lambda a, \lambda b, \lambda c)$. These coordinates are only integer if λ is an integer and, therefore, l is the length of vector (a, b, c). If $\mathbf{u}$ and $\mathbf{v}$ are vectors of the neighbouring sides of the parallelogram with vertices in integer points of the given plane then their *vector product* is a vector with integer coefficients perpendicular to the given plane and the length of this vector is equal to the area of the considered parallelogram. Hence, $S \geq l$.

Now, let us prove that $S \leq l$. To this end it suffices to indicate integer vectors $\mathbf{u}$ and $\mathbf{v}$ lying in the given plane the coordinates of their vector product being equal to (a, b, c). Let d be the greatest common divisor of a and b; $a' = \frac{a}{d}$ and $b' = \frac{b}{d}$; for $\mathbf{u}$ take vector $(-b', a', 0)$. If $\mathbf{v} = (x, y, z)$, then $|\mathbf{u}, \mathbf{v}| = (a'z, b'z, -a'x - b'y)$. Therefore, for z we should take d and select numbers x and y so that $ax+by+cz = 0$, i.e., $-a'x - b'y = c$.

It remains to prove that if p and q are relatively prime then there exist integers x and y such that $px + qy = 1$. Then $px' + qy' = c$ for $x' = cx$ and $y' = cy$. We may assume that $p > q > 0$. Let us successively perform division with a remainder:

$$p = qn_0+r_1,\ q = r_1n_1+r_2,\ r_1 = r_2n_2+r_3,\ \dots,\ r_{k-1} = r_kn_k+r_{k+1},\ r_k = n_{k+1}r_{k+1}.$$

Since numbers p and q are relatively prime, q and r_1 are relatively prime and, therefore, r_1 and r_2 are relatively prime, etc. Hence, r_k and r_{k+1} are relatively prime, i.e., $r_{k+1} = 1$. Let us substitute the value of r_k obtained from the formula $r_{k-2} = r_{k-1}n_{k-1} + r_k$ into $r_{k-1} = r_kn_k + 1$. Then substitute the value of r_{k-1} obtained from the formula $r_{k-3} = r_{k-2}n_{k-2} + r_{k-1}$, etc. At each stage we get a relation of the form $xr_i + yr_{i-1} = 1$ and, therefore, at the end we will get the desired relation.

15.17. Let (x_i, y_i, z_i) be coordinates of the i-th vertex of regular tetrahedron A_1BC_1D. The coordinates of its center which coincides with the center of the cube are $\frac{1}{4}(x_1 + x_2 + x_3 + x_4)$, etc. The first coordinate of the point symmetric to (x_1, y_1, z_1) throuhg the center of the cube is

$$\frac{x_1 + x_2 + x_3 + x_4}{2} - x_1 = \frac{-x_1 + x_2 + x_3 + x_4}{2},$$

and the remaining ones are obtained in a similar fashion. The parity of the number $-x_1 + x_2 + x_3 + x_4$ coincides with that of $x_1 + x_2 + x_3 + x_4$.

Thus we have to prove that numbers $x_1 + x_2 + x_3 + x_4$, etc., are even ones. Let us assume that the origin lies in the fourth vertex of the tetrahedron, i.e., $x_4 = y_4 = z_4 = 0$.

Let u, v, w be integers. It is easy to verify that if $u^2 + v^2 + w^2$ is divisible by 4, then all the numbers u, v and w are even. Therefore, it suffices to verify that $u^2 + v^2 + w^2$, where

$$u = x_1 + x_2 + x_3,\ v = y_1 + y_2 + y_3 \text{ and } w = z_1 + z_2 + z_3$$

is an even number. Let a be the edge of the cube. Since $x_1^2 + y_1^2 + z_1^2 = 2a^2$ and $x_1x_2 + y_1y_2 + z_1z_2 = (\sqrt{2}a)^2 \cos 60° = a^2$, it follows that $u^2 + v^2 + w^2 = 6a^2 + 6a^2 =$

$12a^2$. The number a^2 is an integer because it is the sum of squares of three integer coordinates.

15.18. a) We can assume that one of the vertices of the given parallelepiped is placed in the origin. Let us consider cube K_1 the absolute values of the coordinate of the cube's points do not exceed an integer n. Let us divide the space into parallelepipeds equal to the given one by drawing planes parallel to the faces of the given cube.

The neighbouring parallelepipeds are obtained from each other after a translation by an integer factor and, therefore, all these parallelepipeds have vertices with integer coordinates. Let N be the total number of those of our parallelepipeds that have common points with K_1. All of them lie inside cube K_2 the absolute values of whose coordinates do not exceed $n+d$, where d is the greatest distance between the vertices of the given parallelepiped.

Let us denote the volume of the given parallelepiped by V. Since the considered N parallelepipeds contain K_1 and are contained in K_2, we deduce that $(2n)^3 \leq NV \leq (2n+2d)^3$, i.e.,

$$\left(\frac{1}{2n+2d}\right)^3 \leq \frac{1}{NV} \leq \left(\frac{1}{2n}\right)^3. \tag{1}$$

For each of the considered N parallelepipeds let us write beside its integer points the following numbers: beside any integer point we write number 1, beside any point on the face we write number $\frac{1}{2}$, beside any point on an edge we write number $\frac{1}{4}$ and beside each vertex we write number $\frac{1}{8}$ (as a result, beside points that belong to several parallelepipeds there will be several numbers written). It is easy to verify that the sum of numbers written beside every integer point of K_1 is equal to 1 (we have to take into account that each point on a face belongs to two parallelepipeds a point on an edge belongs to four parallelepipeds and a vertex belongs to eight parallelepipeds); for integer points inside K_2 such a sum does not exceed 1 and for points outside K_2 there are no such points. Therefore, the sum of all the considered numbers is confined between the total number of integer points of cubes K_1 and K_2.

On the other hand, it is equal to $N(1+a+\frac{1}{2}b+\frac{1}{4}c)$. Therefore,

$$(2n+1)^3 \leq N(1+a+\frac{b}{2}+\frac{c}{4}) \leq (2n+2d+1)^3. \tag{2}$$

By multiplying (1) and (2) we see that

$$\left(\frac{2n+1}{2n+2d}\right)^3 \leq \frac{1+a+b/2+c/4}{V} \leq \left(\frac{2n+2d+1}{2n}\right)^3$$

for any positive integer n. Since both the upper and the lower bounds tend to 1 as n tends to infinity,

$$1+a+\frac{b}{2}+\frac{c}{4}=V.$$

b) Let us consider rectangular parallelepiped $ABCDA_1B_1C_1D_1$ whose vertices have integer coordinates, edges are parallel to coordinate axes and the lengths of the edges are equal to 1, 1 and n. Only the vertices are integer points of tetrahedron A_1BC_1D and the volume of this tetrahedron is equal to $\frac{1}{3}n$.

15.19. a) The midpoints of edges of the tetrahedron with edge $2a$ are vertices of an octahedron with edge a. If we cut off this octahedron from the tetrahedron, then there remain 4 tetrahedrons with edge a each.

b) From an octahedron with edge $2a$ we cut off 6 octahedrons with edge a one of the vertices of the cut-off octahedrons being a vertex of the initial octahedron, then there remain 8 tetrahedrons whose bases are triangles formed by the midpoints of the edges of the faces.

15.20. Let us take a regular tetrahedron with edge a and draw planes of its faces and also all the planes parallel to them and distant from them at distance nh, where h is the height of the tetrahedron. Let us prove that these planes divide the space into tetrahedrons and octahedrons with edge a.

Each plane of the tetrahedron's face is divided into equilateral triangles with edge a and there are two types of such triangles: we can identify the triangles of one type with the face of the initial tetrahedron after a translation and we cannot do this with triangles of the other type (see Fig. 110 a)).

Let us prove that any of the considered planes is cut by the remaining planes into equilateral triangles. To this end, it suffices to observe that if the distance of this plane from the plane of a face of the initial tetrahedron is equal to nh, then there exists a regular tetrahedron with edge $(n+1)a$ such that the initial tetrahedron sits at one of the vertices of this larger tetrahedron and our plane is the plane of a face of the tetrahedron that sits at another vertex (see Fig. 110 b)).

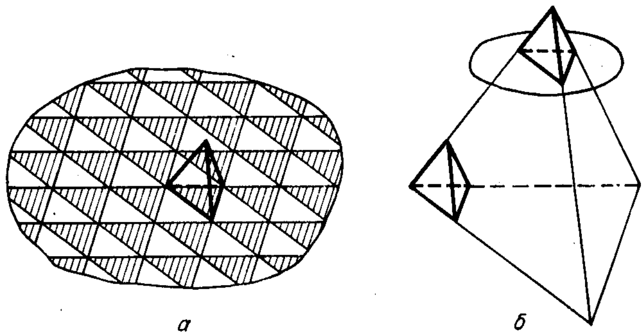

FIGURE 110 (SOL. 15.20)

The translation that sends a vertex of one of these tetrahedrons into a vertex of another one sends the considered system of planes into itself. Any face of any polyhedron into which the space is divided is one of the triangles into which the planes are cut, therefore after one more parallel translation we can either make coincide with the face of the initial tetrahedron or identify a pair of their edges (we assume that the tetrahedron and the polyhedron have a common plane of a face and are situated on one side of it). (????????????)

In the first case the polyhedron is a regular tetrahedron and in the second case it is a regular octahedron (cf. the solution of Problem 15.19 a)).

15.21. For the common vertex of these pyramids take one of the vertices of the cube and for bases three nonadjacent to it faces of the cube.

15.22. If we cut off tetrahedron $A'BC'D$ from cube $ABCDA'B'C'D'$, then the remaining part of the cube splits into 4 tetrahedrons, i.e., a cube can be cut into 5 tetrahedrons.

Let us prove that it is impossible to cut a cube into a lesser number of tetrahedrons. Face $ABCD$ cannot be a face of a tetrahedron into which the cube is cut because at least two tetrahedrons are adjacent to it. Let us consider all the tetrahedrons adjacent to face $ABCD$.

Their heights dropped to this face do not exceed a, where a is the edge of the cube, and the sum of the areas of their faces that lie on $ABCD$ is equal to a^2. Therefore, the sum of their volumes does not exceed $\frac{1}{3}a^3$. Since the faces of one tetrahedron cannot be situated on the opposite faces of the cube, at least 4 tetrahedrons are adjacent to faces $ABCD$ and $A'B'C'D'$, so that the sum of their volumes does not exceed $\frac{2}{3}a^3 < a^3$. Therefore, there is at least one more tetrahedron in the partition.

15.23. The sum of angles of each of the four faces of a tetrahedron is equal to 180° and, therefore, the sum of all the plane angles of a tetrahedron is equal to $4 \cdot 180^\circ$. It follows that the sum of the plane angles at one of the four vertices of the tetrahedron does not exceed 180° and, therefore, the sum of two plane angles at it is less than 180°.

Let, for definiteness, the sum of two plane angles at vertex A of tetrahedron $ABCD$ be less than 180°. On edge AC, take point L and construct in plane ABC angle $\angle ALK$ equal to angle $\angle CAD$. Since

$$\angle KAL + \angle KLA = \angle BAC + \angle CAD < 180^\circ,$$

rays LK and AB intersect and, therefore, we may assume that point K lies on ray AB.

We similarly construct point M on ray AD so that $\angle ALM = \angle BAC$. If point L is sufficiently close to vertex A, points K and M lie on edges AB and AD, respectively. Let us show that plane KLM cuts the tetrahedron in the required way. Indeed, $\triangle KAL = \triangle MLA$ and, therefore, there exists a movement of the space that sends $\triangle KAL$ to $\triangle MLA$. This movement sends tetrahedron $AKLM$ into itself.

15.24. Let us draw all the planes that contain faces of the given polyhedron. All the parts into which they divide the space are convex ones. Therefore, they determine the desired partition.

15.25. a) Inside the polyhedron take an arbitrary point P and cut all its faces into triangles. The triangle pyramids with vertex P whose bases are these triangles give the desired partition.

b) Let us prove the statement by induction on the number of vertices n. For $n = 4$ it is obvious. Let us suppose that it is true for any convex polyhedron with n vertices and prove that then it holds for a polyhedron with $n + 1$ vertices.

Let us select one of the vertices of this polyhedron and cut off it a convex hull of the other n vertices, i.e., the least convex polyhedron that contains them. By inductive hypothesis this convex hull — the convex polyhedron with n vertices — can be divided in the required way.

The remaining part is a polyhedron (perhaps, a nonconvex one) with one fixed point A and the other vertices connected with A by edges. Let us cut its faces into triangles that do not contain vertex A. The triangular pyramids with vertex A whose bases are these triangles give the desired partition.

15.26. The planes of faces of both polyhedrons intersect only along lines that contain their edges. Therefore, each of the parts into which the space is divided has common points with the polyhedron. Moreover, to each vertex, each edge and each face we can assign exactly one part adjacent to it and this will exhaust all the parts except the polyhedron itself. Therefore, the required number is equal to $1+V+F+E$. For the cube it is equal to $1+8+6+12=27$ and for the tetrahedron to $1+4+4+6=15$.

15.27. Denote the number in question by S_n. It is clear that $S_1=2$. Now, let us express S_{n+1} via S_n. To this end let us consider a set of $n+1$ circles on the sphere; select one circle from them . Let the remaining circles divide the sphere into s_n parts ($s_n \leq S_n$). Let the number of parts into which they divide the fixed circle be equal to k.

Since k is equal to the number of the intersection points of the fixed circle with the remaining n circles and any two circles have no more than two points of intersection then $k \leq 2n$. Each of the parts into which the fixed circle is divided divides in halves not more than one of the parts of the sphere obtained earlier. Therefore, the considered $n+1$ circles divide the sphere into not more than $s_n+k \leq S_n+2n$ parts and the equality is attained if any two circles have two common points and no three circles pass through one point. Therefore, $S_{n+1}=S_n+2n$; hence,

$$\begin{aligned} S_n &= S_{n-1}+2(n-1)=S_{n-2}+2(n-2)+2(n-1)=\dots \\ \dots &= S_1+2+4+\dots+2(n-1)=2+n(n-1)=n^2-n+2. \end{aligned}$$

15.28. First, let us prove that n lines no two of which are parallel and no three pass through one point divide the plane into $\frac{n^2+n+2}{2}$ parts. Proof will be carried out by induction on n.

For $n=0$ the statement is obvious. Suppose it is proved for n lines and prove it for $n+1$ lines. Select one line among them. The remaining lines divide it into $n+1$ parts. Each of the lines divides some of the parts into which the plane is divided by n lines into two parts. Therefore, when we draw one line the number of parts increases by $n+1$. It remains to notice that

$$\frac{(n+1)^2+(n+1)+2}{2}=\frac{n^2+n+2}{2}+n+1.$$

For planes the proof is carried out almost in the same way as for lines. We only have to make use of the fact that n planes intersect a fixed plane along n lines, i.e., they are divided into $\frac{n^2+n+1}{2}$ parts.

For $n=0$ the statement is obvious; the identity

$$\frac{(n+1)^3+5(n+1)+6}{6}=\frac{n^3+5n+6}{6}+\frac{n^2+n+2}{2}$$

is subject to a straightforward verification.

15.29. Consider all the intersection points of the given planes. Let us prove that among the given planes there are not more than three planes that do not separate these points. Indeed suppose that there are 4 such planes. No plane can intersect all the edges of tetrahedron $ABCD$ determined by these planes; therefore, the fifth of the given planes (it exists since $n \geq 5$) intersects, for instance, not edge AB itself but its intersection at point F. Let for definiteness sake point B lie between A and F. Then plane BDC separates points A and F; this is impossible.

Therefore, there are $n-3$ planes on either side of which the points under consideration are found. Now, notice that if among all the considered points that lie on one side of one of the given planes we take the nearest one, then the three planes that pass through this point determine together with our plane one of the tetrahedrons to be found.

Indeed, if this tetrahedron were intersected by a plane, then there would be an intersection point situated closer to our plane. Hence, there are $n-3$ planes to each of which at least 2 tetrahedrons are adjacent and to the 3 of the remaining planes at least 1 tetrahedron is adjacent. Since every tetrahedron is adjacent to exactly four planes, the total number of the tetrahedrons is not less than $\frac{1}{4}(2(n-3)+3) = \frac{1}{4}(2n-3)$.

15.30. No, they cannot. Let us divide the plane into triangles equal to the face of the tetrahedron and number them as shown on Fig. 111. Let us cut off a triangle consisting of 4 such triangles and construct a tetrahedron from it.

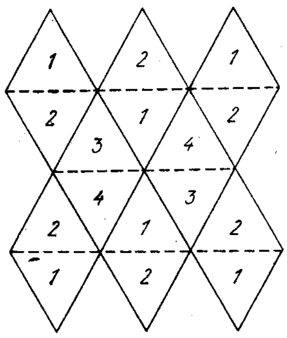

FIGURE 111 (SOL. 15.30)

As is easy to verify that if this tetrahedron is rotated about an edge and then unfolded onto the plane again being cut along the lateral edges, then the number of the triangles of the unfolding coincides with the number of triangles on the plane. Therefore, after any number of rotations of the tetrahedron the numbers of triangles of its unfolding coincide with the number of the tetrahedrons on the plane.

15.31. From the given parallelepiped cut a slice of two cubes thick and glue the remaining parts. Let us prove that the colouring of the new parallelepiped possesses the previous property, i.e., the neighbouring cubes are painted differently. We only have to verify this for cubes adjacent to the planes of $i-th$ cut.

Let us consider four cubes with a common edge adjacent to the plane of the cut and situated on the same side with respect to it. Let them be painted in colours 1–4; let us move in the initial parallelepiped from these cubes to the other plane of the cut. The cubes adjacent to them from the first cut off slice should be painted differently, i.e., colours 5–8.

Further, the small cubes adjacent to this new foursome of cubes are painted not in colours 5-8, i.e., they are painted colours 1–4 and to them in their turn, the cubes painted not colours 1–5, i.e., colours 5–8 are adjacent. Thus, in the new parallelepiped to the considered foursome of small cubes the cubes of other colours are adjacent. Considering all 4 such foursomes for the little cube adjacent to the cut we get the desired statement.

From any rectangular parallelepiped of size $2l \times 2m \times 2n$ we can obtain a cube of size $2 \times 2 \times 2$ with the help of the above-described operation and the little cubes with its corners will be the same as initially. Since any two small cubes of the cube of size $2 \times 2 \times 2$ have at least one common point, all of them are painted differently.

15.32. Let O be the center of the lower base of the cylinder; AB the diameter along which the plane intersects the base; α the angle between the base and the intersecting plane; r the radius of the cylinder. Let us consider an arbitrary generator XY of the cylinder, which has a common point Z with the intersecting plane (point X lies on the lower base). If $\angle AOX = \varphi$, then the distance from point X to line AB is equal to $r \sin \varphi$. Therefore, $XZ = r \sin \varphi \tan \alpha$. It is also clear that $r \tan \alpha = h$, where h is the height of the cylinder.

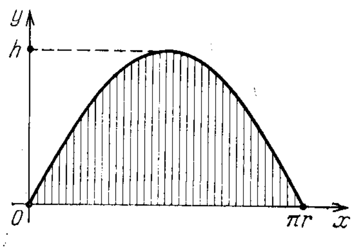

Figure 112 (Sol. 15.32)

Let us unfold the surface of the cylinder to the plane tangent to it at point A. On this plane, introduce a coordinate system selecting for the origin point A and directing Oy-axis upwards parallel to the cylinder's axis. The image of X on the unfolding is $(r\varphi, 0)$ and the image of Z is $(r\varphi, h \sin \varphi)$. Therefore, the unfolding of the surface of the section is bounded by Ox-axis and the graph of the function $y = h \sin(\frac{x}{r})$ (Fig. 112). Its area is equal to

$$\int_0^{\pi r} h \sin(\frac{x}{r}) dx = (-hr \cos(\frac{x}{r}))|_0^{\pi r} = 2hr.$$

It remains to notice that the area of the axial section of the cylinder is also equal to $2hr$.

15.33. First, let us prove that through any two points that lie inside a polyhedron a plane can be drawn that splits the polyhedron into two parts of equal volume.

Indeed, if a plane divides the polyhedron in two parts the ratio of whose volumes is equal to x, then as we rotate this plane through an angle of 180° about the given line the ratio of volumes changes continuously from x to $\frac{1}{x}$. Therefore, at certain moment it becomes equal to 1.

Let us prove the required statement by induction on n. For $n = 1$, draw through two of the three given points a plane that divides the polyhedron into parts of equal volumes. The part to whose interior the third of the given points does not belong is the desired polyhedron.

The inductive step is proved in the same way. Through two of the $3(2^n - 1)$ given points draw a plane that divides the polyhedron into parts of equal volumes. Inside one of such parts there lies not more than $\frac{3(2^n-1)-2}{2} = 3 \cdot 2^{n-1} - 2.5$ points.

Since the number of points is an integer, it does not exceed $3(2^{n-1}-1)$. It remains to apply the inductive hypothesis to the obtained polyhedron.

15.34. Let us consider a parallelepiped for which the given points are vertices and mark its edges that connect given points. Let n be the greatest number of marked edges of this parallelepiped that go out of one vertex; the number n can vary from 0 to 3. An easy case by case checking shows that only variants depicted on Fig. 113 are possible.

Let us calculate the number of parallelepipeds for each of these variants. Any of the four points can be the first, and any of the three remaining ones can be the second one, etc., i.e., we can enumerate 4 points in 24 distinct ways.

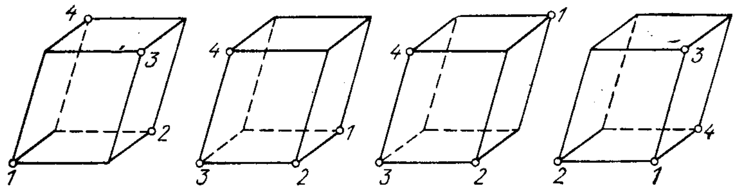

FIGURE 113 (SOL. 15.34)

After the given points are enumerated, then in each of the cases the parallelepiped is uniquely recovered and, therefore, we have to find out which numerations lead to the same parallelepiped.

a) In this case the parallelepiped does not depend on the numeration.

b) Numerations 1, 2, 3, 4 and 4, 3, 2, 1 lead to the same parallelepiped, i.e., there are 12 distinct parallelepipeds altogether.

c) Numerations 1, 2, 3, 4 and 1, 4, 3, 2 lead to the same parallelepiped, i.e., there are 12 distinct parallelepipeds altogether.

d) The parallelepiped only depends on the choice of the first point, i.e., there are 4 distinct parallelepipeds altogether.

As a result we deduce that there are $1+12+12+4=29$ distinct parallelepipeds altogether.

CHAPTER 16. INVERSION AND STEREOGRAPHIC PROJECTION

Let sphere S with center O and radius R in space be given. The *inversion* with respect to S is the transformation that sends an arbitrary point A distinct from O to point A^* that lies on ray OA at the distance $OA^* = \frac{R^2}{OA}$ from point O. The inversion with respect to S will be also called the *inversion with center O and of degree* R^2.

Throughout this chapter the image of point A under an inversion with respect to a sphere is denoted by A^*.

§1. Properties of an inversion

16.1. a) Prove that an inversion with center O sends a plane that passes through O into itself.

b) Prove that an inversion with center O sends a plane that does not contain O into a sphere that passes through O.

c) Prove that an inversion with center O sends a sphere that passes through O into a plane that does not contain point O.

16.2. Prove that an inversion with center O sends a sphere that does not contain point O into a sphere.

16.3. Prove that an inversion sends any line and any circle into either a line or a circle.

The angle between two intersecting spheres (or a sphere and a plane) is the angle between the tangent planes to these spheres (or between the tangent plane and the given plane) drawn through any of the intersection points.

The angle between two intersecting circles in space (or a circle and a line) is the angle between the tangent lines to the circles (or the tangent line and the given line) drawn through any of the intersection points.

16.4. a) Prove that an inversion preserves the angle between intersecting spheres (planes).

b) Prove that an inversion preserves the angle between intersecting circles (lines).

16.5. Let O be the center of inversion, R^2 its degree. Prove that then $A^*B^* = \frac{AB\cdot R^2}{OA\cdot OB}$.

16.6. a) Given a sphere and point O outside it, prove that there exists an inversion with center O that sends the given sphere into itself.

b) Given a sphere and point O inside it, prove that there exists an inversion with center O that sends the given sphere into the sphere symmetric to it with respect to point O.

16.7. Let an inversion with center O send sphere S to sphere S^*. Prove that O is the center of homothety that sends S to S^*.

§2. Let us perform an inversion

16.8. Prove that the angle between circumscribed circles of two faces of a tetrahedron is equal to the angle between the circumscribed circles of two of its other faces.

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16.9. Given a sphere, a circle S on it and a point P outside the sphere. Through point P and every point on the circle S a line is drawn. Prove that the other intersection points of these lines with the sphere lie on a circle.

16.10. Let C be the center of the circle along which the cone with vertex X is tangent to the given sphere. Over what locus points C run when X runs over plane Π that has no common points with the sphere?

16.11. Prove that for an arbitrary tetrahedron there exists a triangle the lengths of whose sides are equal to the products of lengths of the opposite edges of the tetrahedron.

Prove also that the area of this triangle is equal to $6VR$, where V is the volume of the tetrahedron and R the radius of its circumscribed sphere. (*Crelle's formula.*)

16.12. Given a convex polyhedron with six faces all whose faces are quadrilaterals. It is known that 7 of its 8 vertices belong to a sphere. Prove that its 8-th vertex also lies on the sphere.

§3. Tuples of tangent spheres

16.13. Four spheres are tangent to each other pairwise at 6 distinct points. Prove that these 6 points lie on one sphere.

16.14. Given four spheres S_1, S_2, S_3 and S_4 such that spheres S_1 and S_2 are tangent to each other at point A_1; S_2 and S_3 at point A_2; S_3 and S_4 at point A_3; S_4 and S_1 at point A_4. Prove that points A_1, A_2, A_3 and A_4 lie on one circle (or on one line).

16.15. Given n spheres each of which is tangent to all the other ones so that no three of the spheres are tangent at one point, prove that $n \leq 5$.

16.16. Given three pairwise tangent spheres Σ_1, Σ_2, Σ_3 and a tuple of spheres $S_1, S_2, \dots, S_n$ such that each sphere S_i is tangent to spheres Σ_1, Σ_2, Σ_3 and also to S_{i-1} and S_{i+1} (here we mean that $S_0 = S_n$ and $S_{n+1} = S_1$). Prove that if all the tangent points of the spheres are distinct and $n > 2$, then $n = 6$.

16.17. Four spheres are pairwise tangent at distinct points and their centers lie in one plane Π. Sphere S is tangent to all these spheres. Prove that the ratio of the radius of S to the distance from its center to plane Π is equal to $1 : \sqrt{3}$.

16.18. Three pairwise tangent balls are tangent to the plane at three points that lie on a circle of radius R. Prove that there exist two balls tangent to the three given balls and the plane such that if r and ρ ($\rho > r$) are the radii of these balls, then $\frac{1}{r} - \frac{1}{\rho} = \frac{2\sqrt{3}}{R}$.

§4. The stereographic projection

Let plane Π be tangent to sphere S at point A and AB the diameter of the sphere. The *stereographic projection* is the map of sphere S punctured at point B to plane Π under which to point X on the sphere we assign point Y at which ray BX intersects plane Π.

Remark. Sometimes another definition of the stereographic projection is given: instead of plane Π, plane Π' that passes through the center of S parallel to Π is taken. Clearly, if Y' is the intersection point of ray BX with plane Π', then $2\{OY'\} = \{AY\}$ so the difference between these two definitions is enessential.

16.19. a) Prove that the stereographic projection coincides with the restriction to the sphere of an inversion in space.

b) Prove that the stereographic projection sends a circle on the sphere that passes through point B into a line and a circle that does not pass through B into a circle.

c) Prove that the stereographic projection preserves the angles between circles.

16.20. Circle S and point B in space are given. Let A be the projection of point B to a plane that contains S. For every point D on S consider point M — the projection of A to line DB. Prove that all points M lie on one circle.

16.21. Given pyramid $SABCD$ such that its base is a convex quadrilateral $ABCD$ with perpendicular diagonals and the plane of the base is perpendicular to line SO, where O is the intersection point of diagonals, prove that the bases of the perpendiculars dropped from O to the lateral faces of the pyramid lie on one circle.

16.22. Sphere S with diameter AB is tangent to plane Π at point A. Prove that the stereographic projection sends the symmetry through the plane parallel to Π and passing through the center of S into the inversion with center A and degree AB^2. More exactly, if points X_1 and X_2 are symmetric through the indicated plane and Y_1 and Y_2 are the images of points X_1 and X_2 under the stereographic projection, then Y_1 is the image of Y_2 under the indicated inversion.

Solutions

16.1. Let R^2 be the degree of the considered inversion.

a) Consider a ray with the beginning point at O and introduce a coordinate system on the ray. Then the inversion sends the point with coordinate x to the point with coordinate $\frac{R^2}{x}$. Therefore, the inversion preserves a ray with the beginning point at O. It follows that the inversion maps the plane that passes through point O into itself.

b) Let A be the base of the perpendicular dropped from point O to the given plane and X any other point on this plane. It suffices to prove that $\angle OX^*A^* = 90^0$ (indeed, this means that the image of any point of the considered plane lies on the sphere with diameter OA^*). Clearly,

$$OA^* : OX^* = \left(\frac{R^2}{OA}\right) : \left(\frac{R^2}{OX}\right) = OX : OA,$$

i.e., $\triangle OX^*A^* \sim \triangle OAX$. Therefore, $\angle OX^*A^* = \angle OAX = 90^0$. To complete the proof we have to notice that any point Y of the sphere with diameter OA^* distinct from point O is the image of a point of the given plane — the intersection point of ray OY with the given plane.

c) We can carry out the same arguments as in the proof of the preceding heading but even more obviously can use it directly because $(X^*)^* = X$.

16.2. Given sphere S. Let A and B be points at which the line that passes through point O and the center of S intersects S; let X be an arbitrary point of S. It suffices to prove that $\angle A^*X^*B^* = 90^\circ$. From the equalities $OA \cdot OA^* = OX \cdot OX^*$ and $OB \cdot OB^* = OX \cdot OX^*$ it follows that $\triangle OAX \sim \triangle OX^*A^*$ and $\triangle OBX \sim \triangle OX^*B^*$ which, in turn, implies the corresponding relations between oriented angles: $\angle(A^*X^*, OA^*) = \angle(OX, XA)$ and $\angle(OB^*, X^*B^*) = \angle(XB, OX)$. Therefore,

$$\angle(A^*X^*, X^*B^*) = \angle(A^*X^*, OA^*) + \angle(OB^*, X^*B^*) = \\ \angle(OX, XA) + \angle(XB, OX) = \angle XB, XA) = 90^\circ.$$

16.3. It is easy to verify that any line can be represented as the intersection of two planes and any circle as the intersection of a sphere and a plane. In Problems 16.1 and 16.2 we have shown that every inversion sends any plane and any sphere into either a plane or a sphere. Therefore, every inversion sends any line and any circle into a figure which is the intersection of either two planes, or a sphere and a plane, or two spheres. It remains to notice that the intersection of a sphere and a plane (as well as the intersection of two spheres) is a circle.

16.4. a) First, let us prove that every inversion sends tangent spheres to either tangent spheres or to a sphere and a plane tangent to it, or to a pair of parallel planes. This easily follows from the fact that tangent spheres are spheres with only one common point and the fact that under an inversion a sphere turns into a sphere or a plane. Therefore, the angle between the images of spheres is equal to the angle between the images of the tangent planes drawn through the intersection point.

Therefore, it remains to carry out the proof for two intersecting planes Π_1 and Π_2. Under an inversion with center O plane Π_i turns into a sphere that passes through point O and the tangent plane to it at this point is parallel to plane Π_i. This implies that the angle between the images of planes Π_1 and Π_2 is equal to the angle between planes Π_1 and Π_2.

b) First, we have to formulate the definition of the tangency of circles in the form invariant under an inversion. This is not difficult to do: we say that two circles in space are tangent to each other if and only if they belong to one sphere (or plane) and have only one common point. Now it is easy to prove that tangent circles pass under an inversion to tangent circles (a circle and a line) or a pair of parallel lines. The rest of the proof is carried out precisely as in heading a).

16.5. Clearly, $OA \cdot OA^* = R^2 = OB \cdot OB^*$. Therefore, $OA : OB^* = OB : OA^*$, i.e., $\triangle OAB \sim \triangle OB^*A^*$. Hence,

$$\frac{A^*B^*}{AB} = \frac{OB^*}{OA} = \frac{OB^*}{OA} \cdot \frac{OB}{OB} = \frac{R^2}{OA \cdot OB}.$$

16.6. Let X and Y be the intersection points of the given sphere with a line that passes through point O. Let us consider the inversion with center O and coefficient R^2. It is easy to verify that in both headings of the problem we actually have to select the coefficient R^2 so that for any line that passes through O the equality $OX \cdot OY = R^2$ would hold. It remains to notice that the quantity $OX \cdot OY$ does not depend on the choice of the line.

16.7. Let A_1 be a point on sphere S and A_2 be another intersection point of line OA_1 with sphere S (if OA_1 is tangent to S, then $A_2 = A_1$). It is easy to verify that the equality $d = OA_1 \cdot OA_2$ is the same for all the lines that intersect sphere S. If R^2 is the degree of the inversion, then $OA_1^* = \frac{R^2}{OA_1} = \frac{R^2}{d}OA_2$. Therefore, if point O lies inside sphere S, then A_1^* is the image of point A_2 under the homothety with center O and coefficient $\frac{R^2}{d}$ and if point O lies outside S, then A_1^* is the image of A_2 under the homothety with center O and coefficient $\frac{R^2}{d}$.

16.8. Let us apply an inversion with center at vertex D to tetrahedron $ABCD$. The circumscribed circles of faces DAB, DAC and DBC pass to lines A^*B^*, A^*C^* and B^*C^* and the circumscribed circle of face ABC to the circumscribed circle S of triangle $A^*B^*C^*$. Since any inversion preserves the angles between circles (or lines), cf. Problem 16.4 b), we have to prove that the angle between line A^*B^* and circle S is equal to the angle between lines A^*C^* and B^*C^* (Fig. 114). This

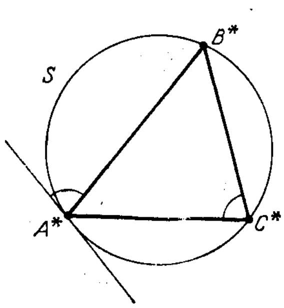

FIGURE 114 (SOL. 16.8)

follows directly from the fact that the angle between the tangent to the circle at point A^* and chord A^*B^* is equal to the inscribed angle $A^*C^*B^*$.

16.9. Let X and Y be the intersection points of the sphere with the line that passes through point P. It is not difficult to see that the quantity $PX \cdot PY$ does not depend on the choice of the line; let us denote it by R^2.

Let us consider the inversion with center P and degree R^2. Then $X^* = Y$. Therefore, the set of the second intersection points with the sphere of the lines that connect P with the points of the circle S is the image of S under this inversion. It remains to notice that the image of a circle under an inversion is a circle.

16.10. Let O be the center of the given sphere, XA a tangent to the sphere. Since AC is a height of right triangle OAX, then $\triangle ACO \sim \triangle XAO$. Hence, $OA : CO = XO : AO$, i.e., $CO \cdot XO = AO^2$. Therefore, point C is the image of point X under the inversion with center O and degree $AO^2 = R^2$, where R is the radius of the given sphere. The image of plane Π under this inversion is the sphere of diameter $\frac{R^2}{OP}$, where P is the base of the perpendicular dropped from point O to plane Π. This sphere passes through point O and its center lies on segment OP.

16.11. Let tetrahedron $ABCD$ be given. Let us consider the inversion with center D and degree r^2. Then

$$A^*B^* = \frac{ABr^2}{DA \cdot DB}, \quad B^*C^* = \frac{BCr^2}{BD \cdot DC} \text{ and } A^*C^* = \frac{ACr^2}{DA \cdot DC}.$$

Therefore, if we take $r^2 = DA \cdot DB \cdot DC$, then $A^*B^*C^*$ is the desired triangle.

To compute the area of triangle $A^*B^*C^*$, let us find the volume of tetrahedron $A^*B^*C^*D$ and its height drawn from vertex D. The circumscribed sphere of tetrahedron $ABCD$ turns under the inversion to plane $A^*B^*C^*$. Therefore, the distance from this plane to point D is equal to $\frac{r^2}{2R}$.

Further, the ratio of volumes of tetrahedrons $ABCD$ and $A^*B^*C^*D$ is equal to the product of ratios of lengths of edges that go out of point D. Therefore,

$$V_{A^*B^*C^*D} = V\frac{DA^*}{DA}\frac{DB^*}{DB}\frac{DC^*}{DC} = V\left(\frac{r}{DA}\right)^2\left(\frac{r}{DB}\right)^2\left(\frac{r}{DC}\right)^2 = Vr^2.$$

Let S be the area of triangle $A^*B^*C^*$. Making use of the formula $V_{A^*B^*C^*D} = \frac{1}{3}h_dS$ we get $Vr^2 = \frac{1}{3}\frac{r^2}{2R}S$, i.e., $S = 6VR$.

16.12. Let $ABCDA_1B_1C_1D_1$ be the given polyhedron where only about vertex C_1 we do not know if it lies on the given sphere (Fig. 115 a)). Let us consider an

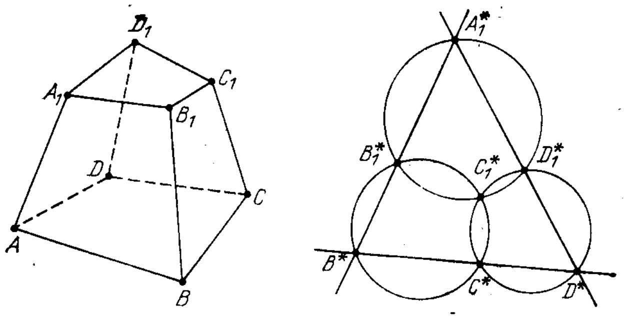

FIGURE 115 (SOL. 16.12)

inversion with center A. This inversion sends the given sphere into a plane and the circumscribed circles of faces $ABCD$, ABB_1A_1 and AA_1D_1D into lines (Fig. 115 b)).

Point C_1 is the intersection point of planes $A_1B_1D_1$, CD_1D and BB_1C, therefore, its image C_1^* is the intersection point of the images of these planes, i.e., the circumscribed spheres of tetrahedrons $AA_1^*B_1^*D_1^*$, $AC^*D_1^*D^*$ and $AB^*B_1^*C^*$ (we have in mind the point distinct from A). Therefore, in order to prove that point C_1 belongs to this sphere it suffices to prove that the circumscribed circles of triangles $A_1^*B_1^*D_1^*$, $C^*D_1^*D^*$ and $B^*B_1^*C^*$ have a common point (see Problem 28.6 a)).

16.13. It suffices to verify that an inversion with the center at the tangent point of two spheres sends the other 5 tangent points into points that lie in one plane. This inversion sends two spheres into a pair of parallel planes and two other spheres into a pair of spheres tangent to each other. The tangent points of these two spheres with planes are vertices of a square and the tangent point of the spheres themselves is the intersection point of the diagonals of the square.

16.14. Let us consider an inversion with center A_1. Spheres S_1 and S_2 turn into parallel planes S_1^* and S_2^*. We have to prove that points A_2^*, A_3^* and A_4^* lie on one line (A_2^* is the tangent point of plane S_2^* and sphere S_3^*, A_3^* the tangent point of spheres S_3^* and S_4^*, A_4^* the tangent point of plane S_1^* and sphere S_4^*).

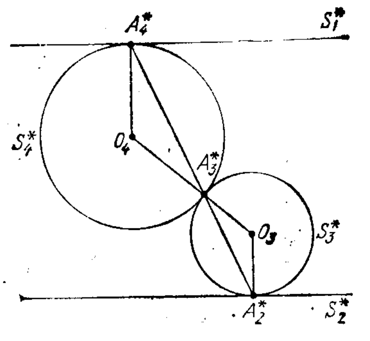

FIGURE 116 (SOL. 16.14)

Let us consider the section with the plane that contains parallel segments $A_2^*O_3$ and $A_4^*O_4$, where O_3 and O_4 are the centers of spheres S_3^* and S_4^* (Fig. 116). Point

A_3^* lies on segment O_3O_4, therefore, it lies in the plane of the section. The angles at vertices O_3 and O_4 of isosceles triangles $A_2^*O_3A_3^*$ and $A_3^*O_4A_4^*$ are equal since $A_2^*O_3 \parallel A_4^*O_4$. Therefore, $\angle O_4A_3^*A_4^* = \angle O_3A_3^*A_2^*$; hence, points A_2^*, A_3^* and A_4^* lie on one line.

16.15. Consider an inversion with the center at one of the tangent points of spheres. These spheres turn into a pair of parallel planes and the remaining $n-2$ spheres into spheres tangent to both these planes. Clearly, the diameter of any sphere tangent to two parallel planes is equal to the distance between the planes.

Now, consider the section with the plane equidistant from the two of our parallel planes. In the section we get a system of $n-2$ pairwise tangent equal circles. It is impossible to place more than 3 equal circles in plane so that they would be pairwise tangent. Therefore, $n-2 \leq 3$, i.e., $n \leq 5$.

16.16. Let us consider an inversion with the center at the tangent point of spheres Σ_1 and Σ_2. The inversion sends them into a pair of parallel planes and the images of the other spheres are tangent to these planes and, therefore, their radii are equal. Thus, in the section with the plane equidistant from these parallel planes we get what is depicted on Fig. 117.

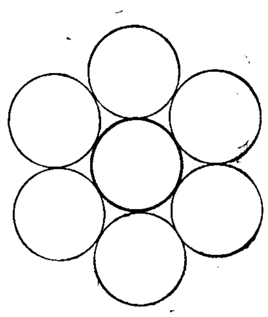

FIGURE 117 (SOL. 16.16)

16.17. Let us consider an inversion with center at the tangent point of certain of two spheres. This inversion sends plane Π into itself because the tangent point of two spheres lies on the line that connects their centers; the spheres tangent at the center of the inversion turn into a pair of parallel planes perpendicular to plane Π, and the remaining two spheres into spheres whose centers lie in plane Π since they were symmetric with respect to it and so they will remain. The images of these spheres and the images of sphere S are tangent to a pair of parallel planes and, therefore, their radii are equal.

For the images under the inversion let us consider their sections with the plane equidistant from the pair of our parallel planes. Let A and B be points that lie in plane Π — the centers of the images of spheres, let C be the center of the third sphere and CD the height of isosceles triangle ABC. If R is the radius of sphere S^*, then $CD = \frac{\sqrt{3}}{2}AC = \sqrt{3}R$. Therefore, for sphere S^* the ratio of the radius to the distance from the center to plane Π is equal to $1 : \sqrt{3}$. It remains to observe that for an inversion with the center that belongs to plane Π the ratio of the radius of the sphere to the distance from its center to plane Π is the same for spheres S and S^*, cf. Problem 16.7.

16.18. Let us consider the inversion of degree $(2R)^2$ with center O at one of the tangent points of the spheres with the plane; this inversion sends the circle that passes through the tangent points of the spheres with the plane in line AB whose distance from point O is equal to $2R$ (here A and B are the images of the tangent points).

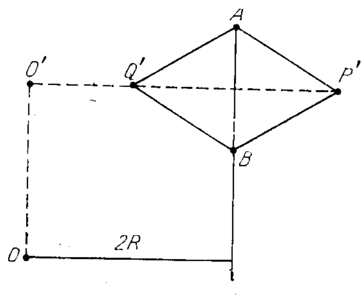

FIGURE 118 (SOL. 16.18)

The existence of two spheres tangent to two parallel planes (the initial plane and the image of one of the spheres) and the images of two other spheres is obvious. Let P and Q be the centers of these spheres, P' and Q' be the projections of points P and O to plane OAB. Then $P'AB$ and $Q'AB$ are equilateral triangles with side $2a$, where a is the radius of spheres, i.e., a half distance between the planes (Fig. 118). Therefore,

$$r = \frac{a \cdot 4R^2}{PO^2 - a^2}, \rho = \frac{a \cdot 4R^2}{QO^2 - a^2}$$

(Problem 16.5), hence,

$$\frac{1}{r} - \frac{1}{\rho} = \frac{PO^2 - QO^2}{4aR^2} = \frac{P'O^2 - Q'O^2}{4aR^2} = \frac{(P'O')^2 - (Q'O')^2}{4aR^2} = \\ = \frac{(2R + \sqrt{3}a)^2 - (2R - \sqrt{3}a)^2}{4aR^2} = \frac{2\sqrt{3}}{R}$$

(here O' is the projection of O to line $P'O'$).

16.19. Let plane Π be tangent to sphere S with diameter AB at point A. Further, let X be a point of S and Y the intersection point of ray BX with plane Π. Then $\triangle AXB \sim \triangle YAB$ and, therefore, $AB : XB = YB : AB$, i.e., $XB \cdot YB = AB^2$. Hence, point Y is the image of X under the inversion with center B and degree AB^2.

Headings b) and c) are corollaries of the just proved statement and the corresponding properties of inversion.

16.20. Since $\angle AMB = 90°$, point M belongs to the sphere with diameter AB. Therefore, point D is the image of point M under the stereographic projection of the sphere with diameter AB to the plane that contains circle S. Therefore, all the points M lie on one circle — the image of S under the inversion with center B and degree AB^2 (cf. Problem 16.19 a)).

16.21. Let us drop perpendicular OA' from point O to face SAB. Let A_1 be the intersection point of lines AB and SA'. Since $AB \perp OS$ and $AB \perp OA'$, plane SOA' is perpendicular to line AB and, therefore, $OA_1 \perp AB$, i.e., A_1 is the projection of point O to side AB. It is also clear that A_1 is the image of point A' under the stereographic projection of the sphere with diameter SO to the plane of the base. Therefore, we have to prove that the projections of point O to sides of quadrilateral $ABCD$ lie on one circle (cf. Problem 2.31).

16.22. Since points X_1 and X_2 are symmetric through the plane perpendicular to segment AB and passing through its center, $\angle ABX_1 = \angle BAX_2$. Therefore, the right triangles ABY_1 and AY_2B are similar. Hence, $AB : AY_1 = AY_2 : AB$, i.e., $AY_1 \cdot AY_2 = AB^2$.

PROBLEMS FOR INDEPENDENT STUDY

1. The lateral faces of a regular n-gonal pyramid are lateral faces ofa regular quadrilateral pyramid. The vertices of the bases of the quadrilateral pyramid distinct from the vertices of the n-gonal pyramid form a regular $2n$-gon. For what n this is possible? Find the dihedral angle at the base of the regular n-gonal pyramid.

2. Let K and M be the midpoints of edges AB and CD of tetrahedron $ABCD$. On rays DK and AM, points L and P, respectively, are taken so that $\frac{DL}{DK}=\frac{AP}{AM}$ and segment LP intersects edge BC. In what ratio the intersection point of segments LP and BC divides BC?

3. Is the sum of areas of two faces of a tetrahedron necessarily greater than the area of a third face?

4. The axes of n cylinders of radius r each lie on one plane. The angles between the neighbouring axes are equal to $2\alpha_1$, $2\alpha_2$, $\dots$, $2\alpha_n$, respectively. Find the volume of the common part of the given cylinders.

5. Is there a tetrahedron such that the areas of three of its faces are equal to 5, 6 and 7 and the radius of the inscribed ball is equal to 1?

6. Find the volume of the greatest regular octahedron inscribed in a cube with edge a.

7. Given tetrahedron $ABCD$. On its edges AB and CD points K and M, respectively, are taken so that $\frac{AK}{KB}=\frac{DM}{MC}\neq 1$. Through points K and M a plane that divides the tetrahedron into two polyhedrons of equal volumes is drawn. In what ratio does this plane divide edge BC?

8. Prove that the intersection of three right circular cylinders of radius 1 whose axes are pairwise perpendicular fits into a ball of radius $\sqrt{\frac{3}{2}}$.

9. Prove that if the opposite sides of a spatial quadrilateral are equal, then its opposite angles are also equal.

10. Let $A'B'C'$ be an orthogonal projection of triangle ABC. Prove that it is possible to cover $A'B'C'$ with triangle ABC.

11. The opposite sides of a spatial hexagon are parallel. Prove that these sides are pairwise equal.

12. What is the area of the smallest face of the tetrahedron whose edges are equal to 6, 7, 8, 9, 10 and 11 and volume is equal to 48?

13. Given 30 nonzero vectors in space, prove that there are two vectors among them the angle between which is smaller than 45°.

14. Prove that there exists a projection of any polyhedron, which is a polygon with the number of vertices not less than 4. Prove also that there exists a projection of the polyhedron, which is a polygon with the number of vertices not more than $n-1$, where n is the number of vertices of the polyhedron.

15. Given finitely many points in space such that the volume of any tetrahedron with the vertices in these points does not exceed 1, prove that all these points can be placed inside a tetrahedron of volume 8.

16. Given a finite set of red and blue great circles on a sphere, prove that there exists a point through which 2 or more circles of one colour and none of the circles of the other colour pass.

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17. Prove that if in a convex polyhedron from each vertex an even number of edges exit, then in any of its section with a plane that does not pass through any of its vertices we get a polygon with an even number of sides.

18. Does an arbitrary polyhedron contain not less than three pairs of faces with the same number of sides?

19. The base of a pyramid is a parallelogram. Prove that if the opposite plane angles of the vertex of the pyramid are equal, then the opposite lateral edges are also equal.

20. On the edges of a polyhedron signs "+" and "−" are placed. Prove that there exists a vertex such that going around it we will encounter the change of sign not oftener than 4 times.

21. Prove that any convex body of volume V can be placed in a rectangular parallelepiped of volume $6V$.

22. Given a unit cube $ABCDA_1B_1C_1D_1$; take points M and K on lines AC_1 and BC, respectively, so that $\angle AKM = 90^\circ$. What is the least value the length of AM can take?

23. A rhombus is given; its the acute angle is equal to α. How many distinct parallelepipeds all whose faces are equal to this rhombus are there? Find the ratio of volumes of the greatest of such parallelepipeds to the smallest one.

24. On the plane, there are given 6 segments equal to the edges of a tetrahedron and it is indicated which edges are neighbouring ones. Construct segments equal to the distance between the opposite edges of the tetrahedron, the radius of the inscribed and the radius of the circumscribed spheres.

Prove that for any n there exists a sphere inside which there are exactly n points with integer coordinates.

26. A polyhedron M' is the image of a convex polyhedron M under the homothety with coefficient $-\frac{1}{3}$. Prove that there exists a parallel translation that sends polyhedron M' inside M. Prove that if the homothety coefficient is $h < -\frac{1}{3}$, then this statement becomes false.

27. Is it possible to form a cube with edge k from black and white unit cubes so that any unit cube has exactly two of its neighbours of the same colour as itself? (Two cubes are considered neighbouring if they have a common face.)

28. Let R be the radius of the sphere circumscribed about tetrahedron $ABCD$. Prove that

$$CD^2 + BC^2 + BD^2 < 4R^2 + AB^2 + AC^2 + AD^2.$$

29. Prove that the perimeter of any section of a tetrahedron does not exceed the greatest of the perimeters of the tetrahedron's faces.

30. On a sphere, n great circles are drawn. They divide the sphere into some parts. Prove that these parts can be painted two colours so that any two neighbouring parts are painted different colours. Moreover, for any odd n the diametrically opposite parts can be painted distinct colours and for any even n they can be painted one colour.

31. Does there exist a convex polyhedron with 1988 vertices such that from no point in space outside the polyhedron it is possible to see all its vertices while it is possible to see any of 1987 of its vertices. (We assume that the polyhedron is not transparent.)

32. Let r be the radius of the ball inscribed in tetrahedron $ABCD$. Prove that

$$r < \frac{AB \cdot CD}{2(AB + CD)}.$$

33. Given a ball and two points A and B outside it. Consider possible tetrahedrons $ABMK$ circumscribed about the given ball. Prove that the sum of the angles of the spatial quadrilateral $AMBK$ is a constant, i.e.,

$$\angle AMB + \angle MBK + \angle BKA + \angle KAM.$$

34. Let positive integers V, E, F satisfy the following relations

$$V - E + F = 2, 4 \leq V \leq \frac{2E}{3} \text{ and } 4 \leq F \leq \frac{2E}{3}.$$

Prove that there exists a convex polyhedron with V vertices, E edges and F faces. (*Euler's formula*.)

35. Prove that it is possible to cut a hole in a regular tetrahedron through which one can move another copy of the undamaged tetrahedron.

36. A cone with vertex P is tangent to a sphere along circle S. The stereographic projection from point A sends S to circle S'. Prove that line AP passes through the center of S'.

37. Given three pairwise skew lines l_1, l_2 and l_3 in space. Consider set M consisting of lines each of which constitutes equal angles with lines l_1, l_2 and l_3 and is equidistant from these lines.

a) What greatest number of lines can be contained in M?

b) If m is the number of lines contained in M, what values can m take?